CYCLES IN WAR

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Abstract

We discuss a simplified version of the well-known card game War in which the cards in the deck have a strict ranking from 1 to n and in which the winning card and losing card are immediately placed, in that order, at the bottom of the winning player's deck. Under this variation of War we show that it is possible for a standard fifty-two card deck to cycle, and we exhibit such a cycle. This result is a special case of a more general result that exhibits a cycle construction for an n-card deck for any value of n that is not a power of 2 or 3 times a power of 2. We also discuss results that show that under some assumptions the types of cycles we exhibit are the only types of cycles that can occur. Finally, we give some open questions related to cycles in War.

1. Introduction

The card game War usually takes so long to play that it often seems as if it will last forever. Is this actually possible, though? Can a War deck enter a cycle and thus truly last forever? That question is the focus of this paper.

War involves a standard deck of fifty-two cards of four suits, each with thirteen denominations, being dealt out, face-down, to two players. A turn, or *skirmish*, entails each player flipping over the card at the top of his deck onto the table. The player with the larger-denominated card takes both of these cards and places them face down at the bottom of his deck. If the players turn over cards with the same denomination (e.g., a queen of hearts and a queen of spades), then the players *battle*: Each player places three more cards on the table and then turns over a fourth card. The player with the larger-denominated fourth card takes all the cards currently on the table. If the fourth cards have the same denomination then battles continue to be held until one player wins a battle, taking all the cards accumulated on the table. The game ends when one player has won all of the cards.

As a first attempt at addressing the question of cycles in War we consider a simplified version of the game in which the fifty-two cards in the deck have a strict ranking from 1 to 52. This makes battles impossible. There are also different options for returning cards that one has won to one's deck: We consider the variation in which the winning card and losing card are immediately placed, in that order, at the bottom of the winning player's deck. Under this variation of War we show that it is possible for a fifty-two card deck to cycle, and we exhibit such a cycle. This result is a special case of a more general result that exhibits a cycle construction for an n-card deck, where $n = m \cdot 2^k$, $k \geq 0$, $m \geq 5$, and m odd. We also have several results that show that under some assumptions the types of cycles we exhibit are the only types of cycles that can occur.

There has been only a little work done analyzing War. Brodie [3] investigates the probability that, with a four-suit deck of $4n$ cards, there are no battles in the first $2n$ skirmishes. In particular, he determines that this probability converges to $e^{-3/2}$ as n approaches infinity. Ben-Naim and Krapivsky [1] consider a variation in which each player draws a card randomly from his deck rather than the card on the top. They primarily consider continuous distributions, and they determine steady-state behavior for the players' decks given initial card value densities. Most of their results are for the case in which each player has an infinite number of cards. However, they do investigate the finite-deck case – both the version in which cards are drawn randomly from the deck and a deterministic version in which the winner places both cards at the bottom of his deck. In both situations they predict that the number of skirmishes required for the game to end is $O(n^2)$, where n is the number of cards in the deck, unless the players start with approximately the same card value densities, in which case the number of skirmishes is $O(n^2 \ln n)$. Their experimental results bear this out. Interestingly enough, their Monte Carlo simulations appear to have missed any cycles in the deterministic case, while our work shows that they can occur. This points to the rarity of cycles. Berlekamp, Conway, and Guy also report in Vol. 4 of *Winning Ways for Your Mathematical Plays* [2, p. 892] that Marc Paulhus has shown that the similar game of Beggar-My-Neighbor can cycle, although the cycles are rare: About 1 in 150,000 games played with the usual 52-card deck cycle. (For more on Beggar-My-Neighbor, see Paulhus [5].) Finally, a paper by Lakshtanov and Roshchina [4] on cycles in War appeared on the arXiv in the final stages of the refereeing process for this paper. They prove that if there are positive probabilities of both the winning card and of the losing card being placed first at the bottom of the winning player's deck then the game – regardless of the number of cards in the deck – cannot cycle.

2. Constructing cycles

A key idea in our approach is to focus on categories of cards rather than individual cards. This turns out to make the analysis simpler. Given a sequence of skirmishes in War divide the cards into categories as follows: Cards that never lose are denoted "A" cards. Cards that only lose to A cards are denoted "B" cards. Similarly, cards that lose only to A or B cards

are denoted "C" cards, cards that only lose to A, B, or C cards are denoted "D" cards, and so forth. Part of the reason this category approach works is that if we replace cards with their category labels we can then play through the sequence of skirmishes using the category rankings instead of the individual card rankings to determine the winning card in each skirmish. Because of the way the category labels are defined this preserves the win-loss pairings for the sequence.

Before proving our first lemma we need to define the following term: A *category cycle* is a distribution of category cards such that after a sequence of skirmishes we reach a distribution of category cards previously held. Since there may be multiple cards in each category this implies that a category distribution may cycle before all cards in the distribution even skirmish.

Lemma 1 *There is a n-card cycle in War if and only if there is an n-card category cycle in War.*

Proof. ($\Rightarrow$) Suppose there is an n-card cycle in War. Place the cards into categories based on the sequence of skirmishes consisting of one iteration of the cycle. Since the category labels preserve the win-loss pairings for the sequence, the category version also cycles.
($\Leftarrow$) Suppose there is an n-card category cycle in War. Suppose there are a cards in category A, b cards in category B, and so forth. Replace the A cards with cards ranked from 1 through a, the B cards with cards ranked from $a + 1$ through $a + b$, and so forth. Since every numbered card in a particular category beats every numbered card in a lower-ranked category the win-loss pairings for the category cycle are preserved in the sequence of numbered cards. However, one complete play through the category cycle, while preserving the category positions, permutes the numbered cards within each category and so may not produce a cycle with the numbered cards. Since there are only a finite number of permutations of numbered cards within each category, though, there are some finite number of iterations of the category cycle that must produce a cycle with the numbered cards.   □

Because of Lemma 1 it is sufficient to consider categories of cards rather than numbered cards when searching for cycles.

We also have the following lemma.

Lemma 2 *In a distribution that cycles there must be at least one category A card in each player's deck.*

Proof. Suppose, without loss of generality, that Player 1 has the highest-ranking card, which must be an A card, and there is no category A card in Player 2's deck. Let k denote the highest-ranking card Player 2 ever has in his deck during the cycle. Since k is not an A card, k must lose at least once. However, Player 2 never has any cards ranked higher than k and thus never wins k back. This contradicts the assumption that Player 2 has card k at some point during a cycle.   □

Theorem 1 *Let $n = m \cdot 2^k$, where $k \geq 0$ and m is odd. If $m \geq 5$ then War with n cards can cycle.*

Proof. By Lemma 2 each player must have an A card. Moreover, in a distribution that cycles each A card in a deck must be followed by the card that it defeated most recently. In addition, A cards cannot skirmish. Thus it is not possible for both players to have an A card simultaneously at the top of their decks, and so one player must have at least one card in front of an A card. Therefore, a deck with four or fewer cards cannot cycle. The following distribution cycles after two skirmishes, and thus the smallest deck size that can cycle is five.

Player 1: AB
Player 2: BAB

This distribution can be extended to produce cycles for any odd deck size n by repeatedly adding AB at the end of the players' decks in an appropriate fashion, so that one player always has exactly one more card than the other.

Now, suppose $n = m \cdot 2^k$, where $k \geq 1$ and m is an odd number five or greater. Create an n-card distribution in the following manner: Start with an m-card cycle. Then expand the card distribution according to the following algorithm:

1. Let $i = 1$.

2. Insert after each card in the current distribution a card of the category ranked immediately below it. For instance, after an A card insert a B card, after a B card insert a C card, and so forth.

3. If $i < k$ then $i = i + 1$ and go to Step 2; otherwise, stop.

For example, for $n = 10$ and $n = 20$ the algorithm produces the following distributions.

Deck size	10 cards	20 cards
Player 1	$ABBC$	$ABBCBCCD$
Player 2	$BCABBC$	$BCCDABBCBCCD$

Since there are k iterations of the algorithm, and the number of cards doubles each iteration, the algorithm outputs a distribution with exactly $m \cdot 2^k = n$ cards. Moreover, since Player 1 and Player 2 alternate winning skirmishes in the original m-card distribution, and each iteration inserts after each card in the current distribution a card of the category ranked immediately below it, after iteration k we have a distribution in which each player

wins 2^k skirmishes in a row. Because of the construction of the distribution any of these 2^k consecutive skirmishes involves the same cards in the same order as any other of the 2^k consecutive skirmishes. Thus the distribution has cycled after Player 1 wins 2^k skirmishes and then Player 2 wins 2^k skirmishes, which is when the second A card in Player 1's deck reaches the front.

Finally, the distribution output by the algorithm has one player with 2^k cards more than the other. While this is not possible, unless n is odd, for a standard deal of the cards (for $k \geq 1$ each player has the same number initially), this distribution can be reached from a standard deal of the cards very quickly. To construct such a deal, start with the distribution output by the algorithm. Take the 2^k extra cards at the end of Player X's deck. From these, place the winning cards in the same order in the front of X's deck and the losing cards in the same order in the front of Y's deck. This will produce a deck in which each player has $n/2$ cards. Player X wins the first 2^{k-1} skirmishes, at which point the distribution is the same as that output by the algorithm.   □

Minor variations of the cycles described in the proof of Theorem 1 are possible. For example, if $n = m \cdot 2^k$, with the same restrictions on m and k as in Theorem 1, cycles can be obtained for decks of size n by having Player 1's deck consist of AB, Player 2's deck consist of B followed by a finite sequence of AB pairs so the total number of A's and B's is m, and then expanding the deck via the algorithm in Theorem 1 k times. The resulting distribution cycles after 2^{k+1} skirmishes.

Finally, Theorem 1 shows that the standard fifty-two-card deck can cycle, if one adds a convention that removes the possibility of battles. The algorithm described in the proof of Theorem 1 gives us the following initial fifty-two-card category distribution that cycles after eight skirmishes:

Player 1: $CDABBCBCCDABBCBCCDABBCBCCD$
Player 2: $BCBCCDABBCBCCDABBCBCCDABBC$

If we say in the case of a tie in denomination that a spade beats a heart, which beats a diamond, which beats a club, then we have an ordering of the cards in the deck from 1 to 52. Replace the A cards with the aces and the kings of spades and hearts; the B cards with the other kings, the queens, the jacks, the tens, the nines, and the eight of spades; the C cards with the other eights, the sevens, sixes, fives, fours, and the three of spades; and the D cards with the other threes and the twos. Then this distribution cycles. (It takes 31,920 skirmishes to do so.)

It is unclear whether, in general, the cycles described in the proof of Theorem 1 extend directly to cycles for decks containing two or more suits. For instance, because of the nature of the construction of cycles in the proof of Theorem 1 the number of B cards divided by the number of A cards has a remainder of 1. Thus, if the number of suits is two or more, and a Theorem 1-style construction is used to attempt to create a cycle, there will have to

be some cards designated "A" cards and some designated "B" cards that have the same denomination. We do not know whether these specific A and B cards can be distributed so that they never skirmish.

3. Types of cycles possible

In this section we give some restrictions on the types of cycles possible. First, though, we introduce an alternative labeling system that allows for the possibility of more than 26 categories of cards. In this system we denote the categories $0, 1, 2, 3, \ldots$, where a card is in category j for $j \geq 1$ if and only if the highest-ranked card to which it loses is in category $j - 1$, and a card is in category 0 if and only if it never loses. We continue to use the A, B, C, etc., labeling system, though, except in the few places we truly need the numerical labels, as we find the alphabetical labels generally easier to work with.

Our most important result on the types of cycles possible is that under some assumptions those described in the proof of Theorem 1 are the only kinds of cycles possible. This result is that the following are equivalent:

1. The players alternate winning m consecutive skirmishes for some $m \geq 1$.

2. The players alternate winning 2^k consecutive skirmishes for some $k \geq 0$.

3. If a card loses it does so to one in the category above it.

4. If a card loses it does so to one in the category above it or two categories above it.

5. The players alternate winning 2^k consecutive skirmishes for some $k \geq 0$, the total number of cards in the deck is congruent to $2^k \bmod 2^{k+1}$, the first of the 2^k consecutive skirmishes is won by an A card, and a card is in category j if and only if it appears i positions, $0 \leq i \leq 2^{k+1} - 1$, behind an A card, where j is the number of 1's in the binary representation of i.

The cycles exhibited in the proof of Theorem 1 have the property in Statement 3: When a card loses it does so to one in the category above it. Thus Statement 5 turns out to be a description of those kinds of cycles, including the claim about where a card appears in the deck behind an A card.

A natural approach to constructing cycles is to look for those in which the players have the same number of cards every m skirmishes, for fixed m. The following result shows that such cycles cannot begin with one of the players playing an A card. This is what we see (although it is not proved) with Theorem 1; every cycle there in which a player has an A card at the top of his deck has the players holding a different number of cards.

Theorem 2 *Let a be a category A card held by Player 1 in some distribution that cycles. Then there exists at least one place in the cycle in which a is at the top of Player 1's deck and Players 1 and 2 hold a different number of cards.*

Proof. Suppose not. Then every time a is at the top of Player 1's deck both players hold the same number of cards. This means that every card must skirmish exactly once from a time a is at the top of Player 1's deck until the next time it is at the top. Therefore, if card y is closer to the top of Player 2's deck than card z when a appears at the top of Player 1's deck, and y and z both win their next skirmishes, then y is closer to the top of Player 2's deck than is z when a returns to the top of Player 1's deck. Call this Property 1.

Fix a distribution that cycles in which a is at the top of Player 1's deck. Because A cards never skirmish each other Player 2 cannot have an A card at the top of his deck in this distribution. Let x be the highest-ranked card that is in front of Player 2's first A card in this distribution. Since x is not an A card, x must lose at least once in the cycle and end up in Player 1's deck. However, initially there are no cards in front of Player 2's first A capable of recapturing x. Furthermore, Property 1, together with the fact that a card that just lost a skirmish is placed at the bottom of a player's deck behind the card that defeated it, implies that Player 2 will never have a card in front of its first A capable of winning x back. Thus when x is recaptured by Player 2 it must appear behind Player 2's first A. But then Property 1 implies that x must remain behind Player 2's first A until it is won back by Player 1. Therefore, x cannot return to a position in Player 2's deck in front of the first A card. Thus the distribution cannot cycle.   □

The remainder of the results in this section build up to showing that under certain conditions the kinds of cycles described in the proof of Theorem 1 are the only kind possible.

Call a card that won its previous skirmish a *winning* card and a card that lost its previous skirmish a *losing* card.

Lemma 3 *For a distribution that cycles, if there are an even number of cards in the deck then winning cards skirmish winning cards and losing cards skirmish losing cards, and if there are an odd number of cards in the deck then winning cards skirmish losing cards.*

Proof. If there are an even number of cards in the deck then the players' decks have the same number of cards initially. After each skirmish the winning player gains one card and the losing player loses one card. Thus both players' decks change parity with each skirmish. Thus the players' decks always have the same parity. Because cards are placed at the bottom of a player's deck two at a time, a winning card placed at the bottom of a player's deck will next skirmish a winning card from the other player's deck if and only if the players' decks currently have the same parity. Thus winning cards skirmish winning cards and losing cards skirmish losing cards.

If there are an odd number of cards in the deck then one player has an even number of cards and the other has an odd number of cards. Since the player's decks switch parity after each skirmish they never have the same parity. The rest of the argument is similar to that in the even case.   □

Theorem 3 characterizes cycles when the number of cards in the deck are odd.

Theorem 3 *For a distribution that cycles the following conditions are equivalent:*

1. *The number of cards in the deck are odd.*

2. *There are only two categories of cards, A and B, and each skirmish is won by an A card.*

3. *The players alternate winning skirmishes.*

Proof. $(1 \Rightarrow 2)$ Suppose a B card wins a skirmish that appears in a cycle. Then this card becomes a winning card for one of the players. Since winning cards only meet the other player's losing cards (by Lemma 3) this B card can never skirmish another A card. Thus it never loses after its first win. But this is a contradiction, as B cards that appear in cycles must lose periodically to A cards. Thus no B card ever wins a skirmish. Thus category B must be the lowest card category. Therefore, there are only two categories of cards, A and B, and each skirmish is won by an A card.

$(2 \Rightarrow 3)$ Two A cards cannot appear next to each other in a player's deck in a cycle, as one of the two consecutive cards must be a losing card, and A cards cannot lose. Thus if each skirmish is won by an A card then a player cannot win two skirmishes in a row. Therefore, the players must alternate winning skirmishes.

$(3 \Rightarrow 1)$ At some point in the cycle Player 1 has an A card at the top of his deck. Since every skirmish places a pair of cards at the bottom of a player's deck Player 1 must have an even number of cards at this point. Also, if the next skirmish is denoted skirmish zero, then Player 1 wins the even-numbered skirmishes, while Player 2 wins the odd-numbered skirmishes. Thus Player 2 has an A card in some odd-numbered position. Since Player 2 has an odd number of cards before this A and an odd number of cards after it, Player 2 currently holds an odd number of cards. Therefore, the total number of cards in the deck must be odd.   □

The following four lemmas are all related, and their proofs are similar, although only Lemmas 5 and 6 are used subsequently.

Lemma 4 *For an even-card distribution that cycles each A card has the property that at some point in the cycle it must be immediately followed by a card that will win its next skirmish.*

Proof. Suppose not. Then there is an A card such that every card that loses to the A card loses its next skirmish as well. Suppose z loses to an A card and that Az skirmishes xy next. By supposition z loses to y, which, by Lemma 3, must have lost its previous skirmish to x, which must lose to A. Then the next skirmish involving Ax will have x losing to some card w, which in its previous skirmish lost to some card v, which must lose to A. Since this pattern continues we have an infinite number of cards, which is a contradiction.   □

Lemma 5 *For any distribution that cycles each A card has the property that at some point in the cycle it must be immediately preceded by a card that will lose its next skirmish.*

Proof. Suppose not. Then there is an A card such that, at every point in the cycle, the card that immediately precedes the A card wins its skirmish. Suppose x precedes the A card. Then x wins its next skirmish versus some card y. The A card wins its next skirmish as well, which means that xyA is now a sequence at the bottom of the player's deck. Then y wins its next skirmish against some card z, which means that that the sequence yzA next appears at the bottom of the player's deck. Since this pattern continues we must have an infinite number of cards, which is not possible.   □

The proofs of Lemmas 6 and 7 are similar to those of Lemmas 5 and 4, respectively, and so we do not give them explicitly.

Lemma 6 *For any distribution that cycles each card in the lowest category has the property that at some point in the cycle it must be immediately followed by a card that will win its next skirmish.*

Lemma 7 *For an even-card distribution that cycles each card in the lowest category has the property that at some point in the cycle it must be immediately preceded by a card that will lose its next skirmish.*

The assumptions in the next four theorems – Theorems 4, 5, 6, and 7 – all lead to cycles of the type constructed in the proof of Theorem 1. We state and prove these theorems, and afterward we discuss their conclusions.

Theorem 4 *Suppose, in a distribution that cycles, the players alternate winning m consecutive skirmishes. Then the first skirmish in such a sequence of consecutive skirmishes is won by an A card, and A cards only appear in the first such skirmish. Furthermore, the total number of cards in the deck is congruent to m mod $2m$.*

Proof. Since each player wins exactly m consecutive skirmishes cards are effectively placed at the bottom of a player's deck in sequences of length $2m$. Because each player wins m

skirmishes and then loses m skirmishes each player's win/loss cycle has a period of $2m$, and thus the position of the card in the sequence of length $2m$ when it is placed at the bottom of a player's deck determines whether it wins or loses its next skirmish. Let $x_0, x_1, \ldots, x_{2m-1}$ denote the sequence of positions of $2m$ cards placed at the bottom of Player 1's deck, and let $z_0, z_1, \ldots, z_{2m-1}$ denote the similar sequence of positions for Player 2. Let $w_0, w_1, \ldots, w_{m-1}$ be the positions in the sequence $x_0, x_1, \ldots, x_{2m-1}$ that are the winning positions (i.e., positions such that if a card appears in one of them it will win its next skirmish), and let $y_0, y_1, \ldots, y_{m-1}$ be the positions in the sequence $z_0, z_1, \ldots, z_{2m-1}$ that are the winning positions. (These may wrap around so that, for example, there is some i such that $w_i = x_{2m-1}$ and $w_{i+1} = x_0$.) Because two cards are placed at the bottom of a player's deck with each skirmish, a card in position w_i (y_i) next appears in position x_{2i} (z_{2i}).

By Lemma 2 Player 1 has at least one A card. Suppose Player 1 has an A card that does not appear in position w_0. Then, as the A card never loses, it must remain in positions that comprise some subset of $\{w_1, w_2, \ldots, w_{m-1}\}$. However, the card immediately preceding a card in any of these positions is also in a winning position and so wins its next skirmish. This contradicts Lemma 5. Therefore, all A cards appear at least once in position w_0. Moreover, because winning and losing are determined by position, this also implies that any card that passes through position w_0 appears in the same set of positions as does an A card and thus can never lose. Therefore, the only cards that appear in position w_0 are A cards. A similar argument holds, of course, for position y_0.

Since cards appearing in positions w_0 and y_0 next appear in positions x_0 and z_0, respectively, we have that the cards in positions x_0 and z_0 are A cards. We now show that $w_0 = x_0$ and $y_0 = z_0$. Let v be a card of the lowest category. Let w_{-1} and y_{-1} be the last losing positions in the sequence of losing positions for Players 1 and 2, respectively (so that they are the positions immediately preceding w_0 and y_0). By Lemma 6 and an argument similar to that for the A cards, v must appear at some point in position w_{-1} or y_{-1}. However, a card that appears in position w_{-1} loses to a card in position y_{m-1} and thus next appears in position z_{2m-1}. Similarly, a card that appears in position y_{-1} loses to a card in position w_{m-1} and thus next appears in position x_{2m-1}. It follows, therefore, that x_{2m-1} and z_{2m-1} cannot both be winning positions. Suppose, without loss of generality, that x_{2m-1} is a losing position. In order to have x_0 be a winning position, then, we must have $w_0 = x_0$. Let u be a card that appears in position y_{-1}. Then, as we have just argued, u next appears in position x_{2m-1}. Because $w_0 = x_0$ this is also position w_{-1}, and therefore u next appears in position z_{2m-1}. Since a card that appears in position x_{2m-1} also appears in position z_{2m-1}, and a card of the lowest category must appear in at least one of x_{2m-1} and z_{2m-1}, we have that z_{2m-1} must be a losing position as well. Therefore, $y_0 = z_0$.

Since every A card eventually appears in position w_0 or y_0, and $w_0 = x_0$, $y_0 = z_0$ means that these positions map to themselves, we have that A cards only appear as the first card in a sequence of m consecutive skirmishes won by one of the players.

At some point in the cycle Player 1 has an A card at the top of his deck. Since each A

card begins a sequence of $2m$ cards Player 1 must have some multiple of $2m$ cards in his deck. Player 1 wins the first m skirmishes beginning with this A card, at which point Player 2 wins the next m skirmishes, beginning with an A card of his. Thus Player 2's deck must consist of m cards, followed by some multiple of $2m$ cards. Thus the total number of cards in the deck can be placed in groups of $2m$ cards with m left over.   □

Theorem 5 *Suppose that, in a distribution that cycles, players alternate winning m consecutive skirmishes. Then $m = 2^k$ for some $k \geq 0$.*

Proof. By Theorem 4 the first skirmish in each sequence of m consecutive wins is won by an A card, and A cards only appear at the beginning of a sequence of m consecutive wins. Thus cards in positions 1 through $m - 1$ behind an A card win their next skirmish, and cards in positions m through $2m - 1$ behind an A card lose their next skirmish. Also, because cards are placed in pairs at the bottom of a player's deck with each skirmish, a card that appears n positions, $n < m$, behind an A card next appears $2n$ positions behind the A card. Let b be a B card. Then b loses to an A card a held by Player 1 at some point in the cycle, after which b is one position behind that A card. Since b's position behind a doubles with each win when it loses next it must be 2^k positions behind a, for some $k \geq 0$. This implies $2^{k-1} < m \leq 2^k$. Moreover, since b is a B card its next loss must be to an A card held by Player 2. Since A cards signify the beginning of a sequence of consecutive wins Player 2 must begin a sequence of m consecutive wins 2^k positions behind a. Since $m > 2^{k-1}$ this must be Player 2's next sequence of consecutive wins. Thus Player 1 wins 2^k consecutive skirmishes, which implies $m = 2^k$.   □

Theorem 6 *Suppose that, in a distribution that cycles, the players alternate winning 2^k consecutive skirmishes, for some $k \geq 0$. Then a card is in category j if and only if it appears i positions, $0 \leq i \leq 2^{k+1} - 1$, behind an A card, where j is the number of 1's in the binary representation of i. Moreover, a card only loses to cards in the category immediately above it.*

Proof. As in the proof of Theorem 5, Theorem 4 implies that A cards appear in the first skirmish in each sequence of consecutive wins, A cards only appear there, cards in positions 1 through $2^k - 1$ behind an A card win their next skirmish, cards in positions 2^k through $2^{k+1} - 1$ behind an A card lose their next skirmish, and a card that appears i positions, $0 \leq i \leq 2^k - 1$, behind an A card next appears $2i$ positions behind the A card. Because there are 2^k consecutive skirmishes won, when a card i positions, $2^k \leq i \leq 2^{k+1} - 1$, behind an A card loses, it does so to a card $i - 2^k$ positions behind an A card held by the other player, at which point it next appears $2(i - 2^k) + 1$ positions behind the other player's A card.

Let x be a card that appears in a position i, $0 \leq i \leq 2^{k+1} - 1$, behind an A card. The difference between the binary representations of i and $2i$ is that the latter has an

additional 0 at the end. The difference between the binary representations of i and $i - 2^k$ for $2^k \leq i \leq 2^{k+1} - 1$ is that the latter does not have the leading 1 that the former does. The difference between the binary representations of $2i$ and $2i + 1$ is that the latter has a 1 at the end, whereas the former has a 0. Thus when x wins a skirmish, its current position and its next position have the same number of 1's in their binary representations. When x loses a skirmish, it does so to a card in a position with one fewer 1's in its binary representation, and x's next position still has the same number of 1's in its binary representation. Therefore, the number of 1's in the binary representations of the positions in which a card appears is constant.

Suppose that, for $0 \leq h \leq j$, a card appears in a position with h 1's in its binary representation if and only if it is in category h. This is true for $j = 0$, as the only position with zero 1's in its binary representation is position 0, and only A cards appear there. Let x be a card that appears in a position with $j + 1$ 1's in its binary representation. When x loses it does so to a card with j 1's in its binary representation. Thus x only loses to cards in category j. Therefore x is in category $j + 1$. Now, suppose x is in category $j + 1$. Then x loses to at least one card (actually, only cards) in category j. But all cards in category j have j 1's in their binary representations. Thus x must have $j + 1$ 1's in its binary representation. This establishes the induction claim.

This argument also implies that when a card loses, it does so to a card one category above it. $\square$

Theorem 7 *Suppose, in a distribution that cycles, a card never loses to another card more than two categories above it. Then the players alternate winning 2^k consecutive skirmishes, for some $k \geq 0$.*

Proof. By Theorem 3 the claim is true for $k = 0$, the case when the number of cards are odd. Assume that the number of cards are even. The remainder of the proof is by induction. The induction hypothesis is as follows:

1. Skirmishes are won 2^{k-1} at a time, not necessarily alternating, with 2^k cards effectively placed at the bottom of a player's deck at a time,

2. Of 2^k such cards, the first 2^{k-1} are won or lost as a group on their next skirmish, and the second 2^{k-1} are won or lost as a group on their next skirmish,

3. A player's first card in a sequence of 2^k such cards skirmishes the first card in such a sequence of the other player's,

4. An A card can only appear in position 0 (the first position) of a sequence of 2^k such cards.

By Lemma 3 these assumptions are true, for $k = 1$, when the number of cards are even. Let w and x be cards in positions 0 and 2^{k-1}, respectively, in a sequence of 2^k cards for one player, and let y and z be the cards that skirmish w and x, respectively, when w and x are in these positions. Thus y and z are in positions 0 and 2^{k-1}, respectively, for the other player. The assumptions imply that both w and x won their immediate prior skirmishes and therefore previously appeared in positions 0 and 2^{k-2} of a sequence of 2^{k-1} cards. Therefore, they appeared in a sequence of 2^k cards in those very positions or in positions 2^{k-1} and $3 \cdot 2^{k-2}$. Either way, they both appeared in even positions, and so they must have won their previous skirmishes as well. Moreover, the distance between their positions is now 2^{k-2} rather than 2^{k-1}. This argument can be continued, with the distance between them halving each time, until one of the cards appears in an odd-numbered position. Because card x is in position 2^{k-1}, it must have appeared in an odd-numbered position $k - 1$ skirmishes previously. As card w appears in position 0, it must not have appeared in an odd-numbered position fewer than k skirmishes previously. Because the distance between the two cards halves with each skirmish back, $k - 1$ skirmishes previously the distance was 1. Thus k skirmishes prior w must have defeated x. Similarly, k skirmishes previously z must have lost to y.

Assume, without loss of generality, that w defeats y when w and x meet y and z. By hypothesis neither x nor y can be more than two categories below w. Thus x is in the category immediately above y, in the same category as y, or in the category immediately below y. In the first two cases x defeats z. Moreover, x cannot be in the same category as z, which means that in the third case z must be two categories below y. Thus x defeats z in this case as well. Therefore, the player who wins the first 2^{k-1} skirmishes in a sequence of 2^k skirmishes wins the second 2^{k-1} skirmishes as well. Thus each player actually wins 2^k consecutive skirmishes, each player effectively places 2^{k+1} cards at the bottom of his deck at a time, and the first and second groups of 2^k cards in such a sequence of 2^{k+1} cards are won or lost as a group.

Our assumptions imply that in such a sequence of 2^{k+1} cards an A card can only appear in position 0 or in position 2^k. However, if an A card appears in position 2^k then its immediate prior position would have been 2^{k-1} in some sequence of 2^k cards, and we just showed that that card must have lost at some point. Thus an A card cannot appear in position 2^k, and A cards can only appear in position 0 of such a sequence of 2^{k+1} cards.

By assumption, a player's card in position 0 in such a sequence of 2^{k+1} cards must skirmish a card of the other player's that is either in position 2^k or in position 0. Suppose the former. Now, suppose a B card appears in position 0 in such a sequence of 2^{k+1} cards. It does not skirmish an A card next, as A cards cannot appear in position 2^k. The B card therefore wins its next skirmish and appears again in position 0. This will continue, with the B card never losing after appearing in position 0. Since any B card must lose to an A card at some point in the cycle this is a contradiction, and therefore B cards cannot appear in position 0. Suppose a C card appears in position 0. It does not skirmish an A card next. Suppose it loses to a B card next. Then the B card has just won a skirmish in position 2^k and thus appears next in position 0. We know that this cannot happen, and thus the C card

must win its skirmish when it appears in position 0. But then it appears again in position 0 and so will continue to do so. As with the B cards, this is a contradiction. A similar argument holds for any other category of card. Thus only A cards can appear in position 0. Since position 0 is the only possible position for an A card we have that an A card appears precisely once every 2^k skirmishes, alternating between players. Therefore, each skirmish won by an A card begins a run of exactly 2^k consecutive wins by the player holding that A card.

If a player's first card in a sequence of 2^{k+1} cards skirmishes the other player's first card, then we have satisfied the hypotheses in the induction step for the next higher value of k. This completes the proof. $\qquad\square$

As we stated at the beginning of this section Theorems 4, 5, 6, and 7 imply that the following are equivalent and describe the kinds of cycles in the proof of Theorem 1.

1. The players alternate winning m consecutive skirmishes for some $m \geq 1$.

2. The players alternate winning 2^k consecutive skirmishes for some $k \geq 0$.

3. If a card loses it does so to one in the category above it.

4. If a card loses it does so to one in the category above it or two categories above it.

5. The players alternate winning 2^k consecutive skirmishes for some $k \geq 0$, the total number of cards in the deck is congruent to $2^k \bmod 2^{k+1}$, the first of the 2^k consecutive skirmishes is won by an A card, and a card is in category j if and only if it appears i positions, $0 \leq i \leq 2^{k+1} - 1$, behind an A card, where j is the number of 1's in the binary representation of i.

We have the following interesting corollary as well, which characterizes the situation in which there are three categories of cards.

Corollary 1 *For a distribution that cycles there are three categories of cards if and only if players alternate winning two consecutive skirmishes. Moreover, each pair of skirmishes involves an A card beating a B card and then a B card beating a C card. Also, the total number of cards is congruent to $2 \bmod 4$.*

Proof. ($\Rightarrow$) Suppose there are three categories of cards: A, B, and C. By Theorem 3 there are an even number of cards in the deck. By Lemma 3, then, winning cards only meet winning cards and losing cards only meet losing cards. As C cards can never win, this implies A cards never skirmish C cards. Thus A cards only defeat B cards, and B cards only defeat C cards. Since cards are placed in pairs at the bottom of a player's deck after each skirmish each player's deck must like look a sequence of pairs, each of which is either AB

or BC (with the top card in the deck possibly the second in a pair). However, an AB pair cannot skirmish an AB pair, nor a BC pair another BC pair. Thus every pair of skirmishes must involve AB vs. BC. Therefore, each player wins at least two skirmishes in a row. When a player wins two skirmishes, though, the four cards that go into the bottom of his deck are the sequence $ABBC$. The third and fourth cards are a BC pair that must skirmish (and thus lose to) an AB pair. Since this must be true throughout a player's deck in order for a cycle to occur a player cannot win more than two skirmishes in a row.

($\Leftarrow$) Suppose players alternate winning two consecutive skirmishes. Then Theorem 6 implies that a player's two wins result in placing the sequence $ABBC$ at the bottom of his deck, and an $ABBC$ sequence for one player skirmishes a $BCAB$ sequence for the other. Thus there are only three categories of cards.

Theorem 4 indicates that the total number of cards in the deck is congruent to 2 mod 4. $\square$

A somewhat more involved but basically similar argument to that in Corollary 1 shows that there are four categories of cards if and only if players alternate winning four consecutive skirmishes. These four skirmishes consist of an A vs. B, then two of B vs. C, and finally a C vs. D, and the total number of cards is congruent to 4 mod 8. Unfortunately, a version of Corollary 1 for five categories would be more difficult to prove (and might not even be true), partly because the number of possible pairs is so much larger and partly because a player is not guaranteed to win two consecutive skirmishes: The possible AD vs. BC matchup, for example, would have one player winning the first skirmish and the other player winning the second.

4. Conclusions

We have shown how to construct cycles for War decks of size n for $n = m \cdot 2^k$, $k \geq 0$, m is odd, and $m \geq 5$. We have also shown that if players alternate winning m consecutive skirmishes or if no card loses to a card more than two categories above it then these types of cycles are the only such cycles possible. In addition, we have characterized cycles in the case where n is odd and where there are exactly two or three categories of cards. We have also discussed a characterization of cycles in which there are exactly four categories of cards.

Other variations of War are possible, too. If the losing card from a skirmish enters the winning player's deck before the losing card, then corresponding versions of the results obtained in the previous sections hold. In particular, the version of Theorem 1 for this variation now produces cycles for any n not a power of 2. In the variation in which there are more than two players it is possible to have cycling distributions of the kind described in the proof of Theorem 1. However, unless the number of players is a power of 2 such distributions are not reachable from a standard deal of the deck of cards.

There are many more questions to be addressed concerning cycles in War, including the following:

1. Are there other types of cycles than those constructed in the proof of Theorem 1?

2. Do cycles exist for deck sizes n, where $n = 2^k$ or $n = 3 \cdot 2^k$, $k \geq 0$?

3. What is the probability that an initial deal of cards eventually enters a cycle? For instance, when $n = 10$, we have that almost 11% of (395,940 of $10! = 3,628,800$) deals do so. (This was obtained via enumeration.)

4. What can we say about cycle structures when there are s suits, $s > 1$, so that battles are possible? A standard deck has $s = 4$.

5. Can a standard deal of the cards produce a cycle of some kind when the number of players is not a power of 2?

6. Are there types of cycles possible with more than two players that do not exist in the two-player case?

7. The proofs of many of the main results given in Section 3 involve messy case analyses. Is there some algebraic structure in which our discussion of cycles in War could be placed that would yield cleaner, simpler proofs (and possibly lead to new insights)?

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