Asymptotic Moments of the Bottleneck Assignment Problem

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One of the most important variants of the standard linear assignment problem is the bottleneck assignment problem. In this paper we give a method by which one can find all of the asymptotic moments of a random bottleneck assignment problem in which costs (independent and identically distributed) are chosen from a wide variety of continuous distributions. Our method is obtained by determining the asymptotic moments of the time to first complete matching in a random bipartite graph process and then transforming those, via a Maclaurin series expansion for the inverse cumulative distribution function, into the desired moments for the bottleneck assignment problem. Our results improve on the previous best-known expression for the expected value of a random bottleneck assignment problem, yield the first results on moments other than the expected value, and produce the first results on the moments for the time to first complete matching in a random bipartite graph process.

Key words: bottleneck assignment problem; random assignment problem; asymptotic analysis; probabilistic analysis; complete matching; degree 1; random bipartite graph

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1. Introduction. The behavior of random assignment problems has been the subject of much study in the last few years. One of the most well-known results, due to Aldous [2], is that if Z_n^* is the optimal value of an $n \times n$ random linear assignment problem with independent and identically distributed (iid) exponential(1) costs then $\lim_{n\to\infty} E[Z_n^*] = \zeta(2) = \pi^2/6$. Many other results on the random linear assignment problem and its variants are summarized in the recent survey paper of Krokhmal and Pardalos [13].

One of the most important of these variations is the bottleneck assignment problem. This problem arises in scenarios in which we want to assign n resources to n tasks in such a way that the maximum of the n assignment costs is minimized. For example, if we have n tasks to assign to n machines, the machines operate in parallel, and we want to minimize the time at which the last task is completed, then we have a bottleneck assignment problem. Formally, the bottleneck assignment problem is defined as follows, where c_{ij} is the cost of assigning resource i to task j:

$$\begin{aligned} \min \max_{1\le i,j\le n} \quad & c_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{n} x_{ij} = 1 \text{ for each } j, j \in \{1,2,\ldots,n\}; \\ & \sum_{j=1}^{n} x_{ij} = 1 \text{ for each } i, i \in \{1,2,\ldots,n\}; \\ & x_{ij} \in \{0,1\} \text{ for all } i,j. \end{aligned}$$

For a summary of major results on and algorithms for solving the bottleneck assignment problem, see Section 6.2 in the text of Burkard, Dell'Amico, and Martello [5, pp. 172–191].

Let c_n^* be the optimal cost for an $n \times n$ bottleneck assignment problem. Pferschy [15] proved that if costs are chosen independently from a continuous distribution having cumulative distribution function F such that $\sup\{x|F(x) < 1\} < \infty$ then $\lim_{n\to\infty} E[c_n^*] = \inf\{x|F(x) > 0\}$. When costs are iid uniform[0,1], Pferschy gave the following results on the rate at which $E[c_n^*]$ approaches 0:

$$\begin{aligned} E[c_n^*] &< 1 - \left[\frac{2}{n(n+2)}\right]^{2/n} \frac{n}{n+2} + \frac{123}{610n}, \; n > 78, \\ E[c_n^*] &\ge 1 - nB\left(n, 1 + \frac{1}{n}\right), \end{aligned}$$

where $B(x,y)$ is the beta function. Asymptotically, these bounds yield

$$\frac{\log n + \gamma}{n} + O\left(\frac{(\log n)^2}{n^2}\right) \leq E[c_n^*] \leq \frac{4\log n}{n} + O\left(\frac{1}{n}\right).$$

In this paper we give a method by which one can determine all of the asymptotic moments – not just the expected value – of a random bottleneck assignment problem in which costs are chosen iid from a variety of continuous distributions. Knowing first and second moments means, of course, that we can also determine variances. Our method applies to those distributions whose inverse cumulative distribution function F^{-1} can be expanded in a Maclaurin series (a Taylor series about 0). Examples of distributions for which this is the case include the uniform, the exponential, the half-normal, and the Pareto. For certain values of the distribution's parameters this can also be done for the beta, χ^2, gamma, Weibull, and log-logistic distributions. For the distributions for which our approach can be applied we have, even when $\sup\{x|F(x) < 1\} = \infty$, that $\lim_{n\to\infty} E[c_n^*] = \inf\{x|F(x) > 0\}$. However, our results also yield an asymptotic rate at which $E[c_n^*]$ approaches $\inf\{x|F(x) > 0\}$. For example, in the case where costs are iid uniform[0,1] our results give

$$E[c_n^*] = \frac{\log n + \log 2 + \gamma}{n} + O\left(\frac{(\log n)^2}{n^{7/5}}\right).$$

Combining this with our results on the second moments yields

$$Var[c_n^*] = \frac{\zeta(2) - 2(\log 2)^2}{n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right).$$

Our method builds on two fundamental properties of the bottleneck assignment problem: 1) its optimal cost is taken by one of the c_{ij}'s, and 2) the optimal cost depends only on the relative rank (from 1 to n^2) of the c_{ij}'s and not on their numerical values [5, p. 172]. Focusing on the rank R of the optimal cost c_n^* among the c_{ij}'s, then, can give insight into the behavior of c_n^*. This is our approach: We find asymptotic expressions for the moments of R and then use them, via the Maclaurin series expansion of F^{-1} for the distribution in question, to find the moments of c_n^*.

We obtain the moments of R through their relationship with the time to first complete matching in a random bipartite graph process. (A *complete matching* in a bipartite graph with $2n$ vertices is a set of n edges such that no two edges are incident on the same vertex.) Suppose we have two vertex sets V_1 and V_2 with $|V_1| = |V_2| = n$. Define a random bipartite graph process $\tilde{B} = (B_t)_0^{n^2}$ in the following manner: B_0 is the empty bipartite graph on V_1 and V_2. For $t \geq 1$, B_t is obtained from B_{t-1} by adding an edge at random between a vertex in V_1 and a vertex in V_2, all new edges being equally likely. Let $\tau(\text{match}; \tilde{B})$ denote the first time for which a graph in the process $\tilde{B}$ has a complete matching. We then have the following.

Lemma 1.1 *Let R be the rank of the optimal cost of a random $n \times n$ bottleneck assignment problem. Let $\tilde{B}$ be a random bipartite graph process on vertex sets V_1 and V_2 with $|V_1| = |V_2| = n$. Then, for any k, $P(R = k) = P(\tau(\text{match}; \tilde{B}) = k)$.*

Proof. There are $n^2!$ distinct rankings for the costs in a random $n \times n$ bottleneck assignment problem, and each ranking is equally likely. There are also $n^2!$ distinct random bipartite graph processes on vertex sets V_1 and V_2, and each bipartite graph process is equally likely. Thus it suffices to show a one-to-one mapping from the set of random $n \times n$ bottleneck assignment problems with distinct cost rankings to the set of random bipartite graph processes on vertex sets V_1 and V_2 such that $R = \tau(\text{match}; \tilde{B})$ holds under the mapping.

Suppose we have a specific $n \times n$ bottleneck assignment problem such that cost c_{ij} has relative rank r_{ij} and that R is the rank of the optimal cost c_n^*. For a random bipartite graph process, let t_{ij} denote the time that edge (i,j) enters the graph. Then let $\tilde{B}$ be the bipartite graph process on V_1 and V_2 such that, for each (i,j) pair, $t_{ij} = r_{ij}$. Clearly this defines a one-to-one mapping. Any feasible solution to a bottleneck assignment problem gives rise to a complete matching under this mapping, and vice versa, such that $\max\{r_{ij}\} = \max\{t_{ij}\}$ for the edges (i,j) included in the feasible solution and matching. Therefore, the smallest value of $\max\{r_{ij}\}$ over all possible feasible solutions must equal the smallest value

of $\max\{t_{ij}\}$ over all possible matchings. But the former is, by definition, R, and the latter is $\tau(\text{match}; \tilde{B})$.
□

(Lemma 1.1 is similar to the ideas behind the class of threshold algorithms used to solve the bottleneck assignment problem [5, p. 174].)

Lemma 1.1 implies that one can determine the moments of R by finding the moments of $\tau(\text{match}; \tilde{B})$, and this is our approach. In fact, most of the work in this paper goes toward finding the moments of $\tau(\text{match}; \tilde{B})$. A quantity related to $\tau(\text{match}; \tilde{B})$ is $\tau(\delta(B) \geq 1; \tilde{B})$, the time when a random bipartite graph process first attains minimum degree 1. Clearly, for any $\tilde{B}$, $\tau(\delta(B) \geq 1; \tilde{B}) \leq \tau(\text{match}; \tilde{B})$, and we find the moments of $\tau(\delta(B) \geq 1; \tilde{B})$ as part of the process of finding the moments of $\tau(\text{match}; \tilde{B})$. The moments (including variances) of these two times in themselves constitute a contribution to the theory of random graph processes. While the probability of a random bipartite graph having a particular minimum degree or a complete matching has been studied extensively (see, for example, Erdős and Rényi [8], Frieze and Pittel [10], and Frieze [9]), as far as we can determine there are no published results on the moments of $\tau(\text{match}; \tilde{B})$ or $\tau(\delta(B) \geq 1; \tilde{B})$. In addition, our work shows that the asymptotically dominant terms in the moments of $\tau(\text{match}; \tilde{B})$ are precisely those of $\tau(\delta(B) \geq 1; \tilde{B})$. This is not too surprising, as Bollobás and Thomason [4] prove that $P(\tau(\text{match}; \tilde{B}) = \tau(\delta(B) \geq 1; \tilde{B})) \to 1$ as $n \to \infty$.

In Section 2 we give several results that we need for the remainder of the paper. In Section 3 we determine the moments of $\tau(\delta(B) \geq 1; \tilde{B})$. In Section 4 we do the rest of the work (beyond that in Section 3) needed to find the moments of $\tau(\text{match}; \tilde{B})$. Finally, Section 5 applies these results to the random bottleneck assignment problem.

2. Preliminaries. There are several results on exact and asymptotic values of various sums that we need for our subsequent work (mainly in determining the moments of $\tau(\delta(B) \geq 1; \tilde{B})$ in Section 3). We list them in this section, giving proofs when necessary.

For many discrete random variables it is easier to calculate the factorial moments than the actual moments. As we shall see, the asymptotically dominant terms of the factorial moments of R are the same as those of the usual moments. The following result is known (see, for example, Stirzaker [18, p. 156]).

Lemma 2.1 *If X is a nonnegative, integer-valued random variable and $p \geq 1$ then $E[X^{\underline{p}}] = \sum_{k=0}^{\infty} pk^{\underline{p-1}} P(X > k)$.*

(Here, and subsequently, we take $0^{\underline{0}} = 1$.)

Lemmas 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7 all involve binomial coefficient sums. The first three are known results on alternating binomial transforms, valid for $n \geq 0$ (see, for example, Spivey [17]). We give proofs for the other three.

Lemma 2.2

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1, & n = 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $\left\{ {n \atop k} \right\}$ be a Stirling subset number (or Stirling number of the second kind).

Lemma 2.3 *If $p \geq 1$ then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^p = \left\{ {p \atop n} \right\} (-1)^n n!.$$

Lemma 2.4

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{k+1} = \frac{1}{n+1}.$$

Lemma 2.5 *For $n \geq m$ and $p \geq 0$,*

$$\sum_{k=0}^{m} \frac{k^{\underline{p}} \binom{m}{k}}{\binom{n}{k}} = \frac{p! m^{\underline{p}} (n+1)}{(n+p+1-m)^{\underline{p+1}}}.$$

PROOF. If $m < p$ then both expressions are 0. Assume, then, that $m \geq p$. Let

$$f(m,n,p) = \sum_{k=0}^{m} \frac{k^{\underline{p}}\binom{m}{k}}{\binom{n}{k}},$$
$$g(m,n,p) = \frac{p! m^{\underline{p}}(n+1)}{(n+p+1-m)^{\underline{p+1}}}.$$

We show that $f(m,n,p)$ and $g(m,n,p)$ satisfy the same boundary conditions and recurrence and therefore must be equal. First, the sum $\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}}$ is known to equal $\frac{n+1}{n+1-m}$ [12, p. 174]. Thus $f(m,n,0) = g(m,n,0)$. Also, we have

$$f(m,n,m) = \frac{m^{\underline{m}}\binom{m}{m}}{\binom{n}{m}} = \frac{m^{\underline{m}} m!(n-m)!}{n!} = \frac{m^{\underline{m}} m!}{n^{\underline{m}}} = \frac{m^{\underline{m}} m!(n+1)}{(n+1)^{\underline{m+1}}} = g(m,n,m).$$

Then, if $n \geq m \geq p \geq 1$,

$$\begin{aligned}
f(m,n,p) &= \sum_{k=0}^{m} \frac{k^{\underline{p}}\binom{m}{k}}{\binom{n}{k}} \\
&= \sum_{k=1}^{m} \frac{(k-1)^{\underline{p-1}}(k-p+p)\frac{m}{k}\binom{m-1}{k-1}}{\frac{n}{k}\binom{n-1}{k-1}} \quad [12, \text{ p. } 174] \\
&= \frac{m}{n}\left(\sum_{k=1}^{m} \frac{(k-1)^{\underline{p}}\binom{m-1}{k-1}}{\binom{n-1}{k-1}} + p\sum_{k=1}^{m} \frac{(k-1)^{\underline{p-1}}\binom{m-1}{k-1}}{\binom{n-1}{k-1}}\right) \\
&= \frac{m}{n}\left(\sum_{k=0}^{m-1} \frac{k^{\underline{p}}\binom{m-1}{k}}{\binom{n-1}{k}} + p\sum_{k=0}^{m-1} \frac{k^{\underline{p-1}}\binom{m-1}{k}}{\binom{n-1}{k}}\right) \\
&= \frac{m}{n}\left(f(m-1,n-1,p) + pf(m-1,n-1,p-1)\right).
\end{aligned}$$

Also,

$$\begin{aligned}
g(m,n,p) &= \frac{p! m^{\underline{p}}(n+1)}{(n+p+1-m)^{\underline{p+1}}} \\
&= p!\left(\frac{(m-p)m^{\underline{p}}}{(n+p+1-m)^{\underline{p+1}}} + \frac{(n+p+1-m)m^{\underline{p}}}{(n+p+1-m)^{\underline{p+1}}}\right) \\
&= p!\left(\frac{m^{\underline{p+1}}}{(n+p+1-m)^{\underline{p+1}}} + \frac{m^{\underline{p}}}{(n+p-m)^{\underline{p}}}\right) \\
&= \frac{m}{n}\left(\frac{p!(m-1)^{\underline{p}} n}{(n+p+1-m)^{\underline{p+1}}} + \frac{p(p-1)!(m-1)^{\underline{p-1}} n}{(n+p-m)^{\underline{p}}}\right) \\
&= \frac{m}{n}\left(g(m-1,n-1,p) + pg(m-1,n-1,p-1)\right).
\end{aligned}$$

□

LEMMA 2.6 *For $n \geq m$ and $p \geq 1$,*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{m^{\underline{p-1-k}}(n+1)}{(n+p-k-m)^{\underline{p-k}}} = \frac{(n+p)^{\underline{p}}}{(n+p-m)^{\underline{p}}}.$$

PROOF.

$$\begin{aligned}
\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{m^{\underline{p-1-k}}(n+1)}{(n+p-k-m)^{\underline{p-k}}} &= (n+1)\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{m^{\underline{p-1-k}}(n+p-m)^{\underline{k}}}{(n+p-m)^{\underline{p}}} \\
&= (n+1)\frac{(n+p)^{\underline{p-1}}}{(n+p-m)^{\underline{p}}}
\end{aligned}$$

$$= \frac{(n+p)^{\underline{p}}}{(n+p-m)^{\underline{p}}},$$

where the second-to-last step follows from the binomial theorem for falling factorial powers [12, p. 245]. □

Lemma 2.7 *For $p \geq 1$,*

$$\frac{(p-1)!}{(n+p)^{\underline{p}}} = \sum_{k=1}^{p} \frac{(-1)^{k-1}\binom{p-1}{k-1}}{n+k}$$

Proof. Applying partial fractions decomposition to express the left-hand side as $\sum_{k=1}^{p} \frac{a_k}{n+k}$ we obtain

$$\begin{aligned} a_k &= \frac{(p-1)!}{\prod_{j=1}^{k-1}(-j)\prod_{j=1}^{p-k} j} \\ &= (-1)^{k-1}\frac{(p-1)!}{(k-1)!(p-k)!} \\ &= (-1)^{k-1}\binom{p-1}{k-1}. \end{aligned}$$

□

The following result for falling factorial powers is well-known (see, for example, [12, p. 53]).

Lemma 2.8 *If $p \neq -1$,*

$$\sum_{k=a}^{b} k^{\underline{p}} = \frac{(b+1)^{\underline{p+1}}}{p+1} - \frac{a^{\underline{p+1}}}{p+1}.$$

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$, the nth harmonic number. The asymptotic expression for H_n is also well-known (see, for example, [12, p. 452]).

Lemma 2.9

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right).$$

We then have the following.

Lemma 2.10

$$\sum_{i_1=1}^{n}\sum_{i_2=i_1+1}^{n}\cdots\sum_{i_k=i_{k-1}+1}^{n}\frac{1}{i_1 i_2\cdots i_k} = \frac{1}{k!}(\log n)^k + \frac{\gamma}{(k-1)!}(\log n)^{k-1} + O\left((\log n)^{k-2}\right).$$

Proof. Since there are $k!$ ways to order $i_1, i_2, \ldots, i_k$, we have

$$\sum_{i_1=1}^{n}\sum_{i_2=i_1+1}^{n}\cdots\sum_{i_k=i_{k-1}+1}^{n}\frac{1}{i_1 i_2\cdots i_k} = \frac{1}{k!}\sum_{i_1=1}^{n}\sum_{i_2=1, i_2\neq i_1}^{n}\cdots\sum_{i_k=1, i_k\notin\{i_1,i_2,\ldots,i_{k-1}\}}^{n}\frac{1}{i_1 i_2\cdots i_k}.$$

The expression on the right is bounded above by

$$\frac{1}{k!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{i_1 i_2\cdots i_k} = \frac{(H_n)^k}{k!}$$

and below by

$$\begin{aligned} &\frac{1}{k!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{i_1 i_2\cdots i_k} - \frac{\binom{k}{2}}{k!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_{k-2}=1}^{n}\sum_{i_{k-1}=1}^{n}\frac{1}{i_1 i_2\cdots i_{k-2} i_{k-1}^2} \\ &= \frac{(H_n)^k}{k!} - O\left((H_n)^{k-2}\right). \end{aligned}$$

The result then follows from Lemma 2.9. □

We also need this result.

Lemma 2.11 *For $p \geq 1$,*

$$\sum_{k=1}^{n} \frac{(\log k)^p}{k} = \frac{1}{p+1}(\log n)^{p+1} + O\left(\frac{(\log n)^p}{n}\right).$$

Proof. By the Euler-Maclaurin summation formula [12, p. 474],

$$\begin{aligned}\sum_{k=1}^{n} \frac{(\log k)^p}{k} &= \int_1^n \frac{(\log x)^p}{x} dx + \frac{(\log n)^p}{2n} + \frac{1}{12}\left(\frac{p(\log n)^{p-1} - (\log n)^p}{n^2}\right) + O\left(\frac{(\log n)^p}{n^2}\right) \\ &= \frac{1}{p+1}(\log n)^{p+1} + O\left(\frac{(\log n)^p}{n}\right).\end{aligned}$$

□

Finally, we require the following three asymptotic alternating sum values.

Lemma 2.12

$$\sum_{k=1}^{n}(-1)^{k+1}k = O(n).$$

Proof. Because $\sum_{k=1}^{n}(-1)^{k+1}k = \sum_{k=1}^{n-1}(-1)^{k+1}k + O(n)$, it suffices to prove the result for the case when n is even. We have $\sum_{k=1}^{n}(-1)^{k+1}k = (1-2)+(3-4)+\cdots+(n-1-n) = -\frac{n}{2} = O(n)$. □

Lemma 2.13

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k} = \log 2 + O\left(\frac{1}{n}\right).$$

Proof. It is well-known that $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \log 2$. Because this is an alternating series whose terms decrease in absolute value, truncating it after the nth term produces an error no larger than the order of that of the $(n+1)$th term. □

Lemma 2.14

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{\log k}{k} = \frac{1}{2}(\log 2)^2 - \gamma \log 2 + O\left(\frac{\log n}{n}\right).$$

Proof. As with Lemma 2.12, it suffices to prove the result for the case when n is even. We have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k+1}\frac{\log k}{k} &= \sum_{k=1}^{n}\frac{\log k}{k} - \sum_{k=1}^{n/2}\frac{2\log(2k)}{2k} \\ &= \sum_{k=1}^{n}\frac{\log k}{k} - \sum_{k=1}^{n/2}\frac{\log k}{k} - \sum_{k=1}^{n/2}\frac{\log 2}{k} \\ &= \frac{1}{2}(\log n)^2 + O\left(\frac{\log n}{n}\right) - \frac{1}{2}(\log(n/2))^2 + O\left(\frac{\log n}{n}\right) - \log 2 H_{n/2}, \\ &\quad \text{by Lemma 2.11} \\ &= \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log n)^2 + \log 2 \log n - \frac{1}{2}(\log 2)^2 + O\left(\frac{\log n}{n}\right) \\ &\quad - \log 2\left(\log(n/2) + \gamma + O\left(\frac{1}{n}\right)\right), \text{by Lemma 2.9} \\ &= \log 2 \log n - \frac{1}{2}(\log 2)^2 - \log 2\,(\log n - \log 2 + \gamma) + O\left(\frac{\log n}{n}\right) \\ &= \frac{1}{2}(\log 2)^2 - \gamma \log 2 + O\left(\frac{\log n}{n}\right).\end{aligned}$$

□

3. Moments for time to minimum degree 1. In this section we determine asymptotic results for the moments of $\tau(\delta(B) \geq 1; \tilde{B})$. We have the following:

Theorem 3.1

$$E\left[\tau^p(\delta(B) \geq 1; \tilde{B})\right] = \begin{cases} n \log n + (\log 2 + \gamma)n + O\left(n^{3/5} (\log n)^2\right), & p = 1; \\ n^2 (\log n)^2 + 2(\log 2 + \gamma) n^2 \log n \\ \quad + \left(\zeta(2) + \gamma^2 + 2\gamma \log 2 - (\log 2)^2\right) n^2 + O\left(n^{5/3} (\log n)^2\right), & p = 2; \\ n^p (\log n)^p + p(\log 2 + \gamma) n^p (\log n)^{p-1} + O\left(n^p (\log n)^{p-2}\right), & p \geq 3. \end{cases}$$

Instead of starting the derivation by working directly with $E[\tau^p(\delta(B) \geq 1; \tilde{B})]$ it turns out to be easier to begin with $E\left[p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \tau^{\underline{p-y}}(\delta(B) \geq 1; \tilde{B})\right]$. The largest power of $\tau^{\underline{p-y}}(\delta(B) \geq 1; \tilde{B})$ occurs when $y = 0$, $p! \frac{\binom{p-1}{y}}{(p-y)!} = 1$ when $y = 0$, and the largest term in $\tau^{\underline{p}}(\delta(B) \geq 1; \tilde{B})$ is $\tau^p(\delta(B) \geq 1; \tilde{B})$. Thus the dominant term in $p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \tau^{\underline{p-y}}(\delta(B) \geq 1; \tilde{B})$ is $\tau^p(\delta(B) \geq 1; \tilde{B})$. As we shall see, both expressions have the same dominant term asymptotically as well.

Proof. By Lemma 2.1,

$$E\left[\tau^{\underline{p}}(\delta(B) \geq 1; \tilde{B})\right] = \sum_{m=0}^{n^2} p m^{\underline{p-1}} P\left(\tau(\delta(B) \geq 1; \tilde{B}) > m\right).$$

Let $B_{n,m}$ be a graph chosen at random from the set of all bipartite graphs on V_1 and V_2 with $|V_1| = |V_2| = n$ that contain exactly m edges. Because each random bipartite graph process is equally likely, $P(\tau(\delta(B) \geq 1; \tilde{B}) > m) = P(\delta(B_{n,m}) = 0)$. Riordan and Stein [16] show that, for $n \geq 1$, the number of bipartite graphs with n vertices in each vertex set and m edges having minimum degree one is

$$\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} \binom{ij}{m}.$$

(If $m = 0$ and $n \geq 1$, this expression evaluates to 0, which is of course correct.) As there are $\binom{n^2}{m}$ bipartite graphs with n vertices in each vertex set and m edges, we have

$$P(\delta(B_{n,m}) = 0) = 1 - \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} \binom{ij}{m}}{\binom{n^2}{m}}.$$

Therefore,

$$\begin{aligned} & p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} E\left[\tau^{\underline{p-y}}(\delta(B) \geq 1; \tilde{B})\right] \\ &= p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \sum_{m=0}^{n^2} (p-y) m^{\underline{p-1-y}} \left(1 - \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} \binom{ij}{m}}{\binom{n^2}{m}}\right) \\ &= p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \sum_{m=0}^{n^2} (p-y) m^{\underline{p-1-y}} \qquad (1) \\ &\quad - p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \sum_{m=0}^{n^2} (p-y) m^{\underline{p-1-y}} \left(\frac{\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} \binom{ij}{m}}{\binom{n^2}{m}}\right). \qquad (2) \end{aligned}$$

We now deal with Expressions (1) and (2) separately. Expression (1) is

$$\begin{aligned} p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \sum_{m=0}^{n^2} (p-y) m^{\underline{p-1-y}} &= p! \sum_{y=0}^{p-1} \binom{p-1}{y} \frac{(n^2+1)^{\underline{p-y}}}{(p-y)!}, \text{ by Lemma 2.8} \\ &= p! \sum_{y=0}^{p-1} \binom{p-1}{y} \frac{(n^2+1)!}{(p-y)!(n^2+1-p+y)!} \end{aligned}$$

$$
\begin{aligned}
&= p! \sum_{y=0}^{p-1} \binom{p-1}{y} \binom{n^2+1}{p-y} \\
&= p! \binom{n^2+p}{p}, \text{ by Vandermonde's convolution [12, p. 174]} \\
&= (n^2+p)^{\underline{p}}. \qquad (3)
\end{aligned}
$$

Expression (2) is

$$
\begin{aligned}
&- p! \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-y)!} \sum_{m=0}^{n^2} (p-y) m^{\underline{p-1-y}} \left(\frac{\sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j}\binom{ij}{m}}{\binom{n^2}{m}} \right) \\
&= -p! \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j} \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-1-y)!} \sum_{m=0}^{n^2} \frac{m^{\underline{p-1-y}}\binom{ij}{m}}{\binom{n^2}{m}} \\
&= -p! \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j} \sum_{y=0}^{p-1} \frac{\binom{p-1}{y}}{(p-1-y)!} \frac{(p-1-y)!(ij)^{\underline{p-1-y}}(n^2+1)}{(n^2+p-y-ij)^{\underline{p-y}}}, \\
&\quad \text{by Lemma 2.5} \\
&= -p! \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j} \sum_{y=0}^{p-1} \binom{p-1}{y} \frac{(ij)^{\underline{p-1-y}}(n^2+1)}{(n^2+p-y-ij)^{\underline{p-y}}} \\
&= -p! \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j} \frac{(n^2+p)^{\underline{p}}}{(n^2+p-ij)^{\underline{p}}}, \text{ by Lemma 2.6} \\
&= -p(n^2+p)^{\underline{p}} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j} \frac{(p-1)!}{(n^2+p-ij)^{\underline{p}}} \\
&= -p(n^2+p)^{\underline{p}} \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i}\binom{n}{j} \sum_{q=1}^{p} \frac{(-1)^{q-1}\binom{p-1}{q-1}}{n^2+q-ij}, \text{ by Lemma 2.7} \\
&= -p(n^2+p)^{\underline{p}} \sum_{q=1}^{p} (-1)^{q-1} \binom{p-1}{q-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{n^2+q-ij} \\
&= -p(n^2+p)^{\underline{p}} \sum_{q=1}^{p} (-1)^{q-1} \binom{p-1}{q-1} \sum_{i=1}^{n} (-1)^i \binom{n}{i} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{-\frac{1}{i}}{-\frac{n^2+q}{i}+j}, \\
&\quad \text{by Lemma 2.2.}
\end{aligned}
$$

Let

$$
f(n,q) = \sum_{i=1}^{n} (-1)^i \binom{n}{i} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{-\frac{1}{i}}{-\frac{n^2+q}{i}+j}.
$$

We have

$$
\begin{aligned}
f(n,q) &= \sum_{i=1}^{n} (-1)^i \binom{n}{i} \frac{-n!}{i} \frac{1}{(-\frac{n^2+q}{i}+n)^{\underline{n+1}}}, \text{ by Lemma 2.7} \\
&= \sum_{i=1}^{n} (-1)^i \binom{n}{i} \frac{-n!}{i} \prod_{k=0}^{n} \frac{1}{-\frac{n^2+q}{i}+n-k} \\
&= \frac{n!}{n^2+q} \sum_{i=1}^{n} (-1)^i \binom{n}{i} \prod_{k=0}^{n-1} \frac{1}{-\frac{n^2+q}{i}+n-k} \\
&= \frac{n!}{n^2+q} \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n}{n-i} \prod_{k=0}^{n-1} \frac{-1}{\frac{n^2+q}{n-i}-(n-k)}
\end{aligned}
$$

$$= \frac{n!}{n^2+q} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \prod_{k=0}^{n-1} \frac{1}{\frac{n^2+q}{n-i} - (n-k)}.$$

Now, let

$$S_i = \frac{n!}{n^2+q} \binom{n}{i} \prod_{k=0}^{n-1} \frac{1}{\frac{n^2+q}{n-i} - (n-k)}.$$

Claim 1. $S_{i-1}/S_i > 1$ for $1 \leq i \leq n-1$.

Proof. We have

$$\begin{aligned}
\frac{S_{i-1}}{S_i} &= \frac{\binom{n}{i-1} \prod_{k=0}^{n-1} \frac{1}{\frac{n^2+q}{n-i+1} - (n-k)}}{\binom{n}{i} \prod_{k=0}^{n-1} \frac{1}{\frac{n^2+q}{n-i} - (n-k)}} \\
&= \frac{n!\,(n-i)!\,i!}{(n-i+1)!\,(i-1)!\,n!} \prod_{k=0}^{n-1} \frac{\frac{n^2+q}{n-i} - (n-k)}{\frac{n^2+q}{n-i+1} - (n-k)} \\
&= \frac{i}{n-i+1} \prod_{k=0}^{n-1} \frac{\big(n^2+q-(n-k)(n-i)\big)(n-i+1)}{\big(n^2+q-(n-k)(n-i+1)\big)(n-i)}.
\end{aligned}$$

Let

$$T_{n,i,k} = \frac{\big(n^2+q-(n-k)(n-i)\big)(n-i+1)}{\big(n^2+q-(n-k)(n-i+1)\big)(n-i)}.$$

We claim $T_{n,i,k} > \frac{k+i+q/n}{k+i-1+q/n}$. We have

$$T_{n,i,k} = \frac{(k+i)n^2 + (k+i+q-2ik-i^2)n + i^2k - iq - ik + q}{(k+i-1)n^2 + (k+i+q-2ik-i^2)n + i^2k - iq - ik}.$$

Now, let $g(n,i,k) = (k+i-2ik-i^2)n + i^2k - iq - ik$. The expression $g(n,i,k)$ appears in both the numerator and the denominator of $T_{n,i,k}$ and is clearly negative. Because $(k+i)n^2 + qn + q > (k+i-1)n^2 + qn$, and adding the same negative number to the numerator and denominator of a fraction greater than 1 increases its value (provided both numerator and denominator remain positive), we have

$$T_{n,i,k} > \frac{(k+i)n^2+qn+q}{(k+i-1)n^2+qn} = \frac{k+i+q/n}{k+i-1+q/n} + \frac{q}{(k+i-1)n^2+qn} > \frac{k+i+q/n}{k+i-1+q/n}.$$

Therefore,

$$\frac{S_{i-1}}{S_i} > \frac{i}{n-i+1} \prod_{k=0}^{n-1} \frac{k+i+q/n}{k+i-1+q/n} = \frac{i}{n-i+1} \, \frac{n+i-1+q/n}{i-1+q/n} > 1.$$

□

Since we have an alternating sum in our expression for $f(n,q)$, Claim 1 implies that, for any $k \geq 0$,

$$\sum_{i=0}^{2k+1} (-1)^i S_i \leq f(n,q) \leq \sum_{i=0}^{2k} (-1)^i S_i.$$

To obtain a more precise estimate for $f(n,q)$ we now take a closer look at the S_i's.

$$\begin{aligned}
S_i &= \frac{n!}{n^2+q} \binom{n}{i} \prod_{k=0}^{n-1} \frac{1}{\frac{n^2+q}{n-i} - (n-k)} \\
&= \frac{\Gamma(n+1)\,n!}{(n^2+q)i!\,(n-i)!} \prod_{k=0}^{n-1} \frac{1}{i + \frac{i^2+q}{n-i} + k} \\
&= \frac{n!\,\Gamma(n+1)\Gamma\left(i + \frac{i^2+q}{n-i}\right)}{(n^2+q)i!\,(n-i)!\,\Gamma\left(n+i+\frac{i^2+q}{n-i}\right)}. \qquad (4)
\end{aligned}$$

Alternatively, we can express S_i in terms of the Beta function $B(x,y)$:

$$\begin{aligned} S_i &= \frac{n}{n^2+q}\binom{n}{i} B\left(n, i+\frac{i^2+q}{n-i}\right) \\ &= \frac{n}{n^2+q}\binom{n}{i} B\left(n, \frac{ni+q}{n-i}\right). \end{aligned}$$

For $i=0$ Expression (4) simplifies to

$$S_0 = \frac{\Gamma(n+1)\Gamma\left(\frac{q}{n}\right)}{(n^2+q)\Gamma\left(n+\frac{q}{n}\right)} = \frac{n\Gamma(n)\Gamma\left(1+\frac{q}{n}\right)\left(n+\frac{q}{n}\right)}{(n^2+q)\frac{q}{n}\Gamma\left(n+1+\frac{q}{n}\right)} = \frac{n\Gamma(n)\Gamma\left(1+\frac{q}{n}\right)}{q\Gamma\left(n+1+\frac{q}{n}\right)},$$

or, in terms of the Beta function,

$$S_0 = \frac{n}{q} B\left(n, 1+\frac{q}{n}\right).$$

We now obtain an asymptotic expression for S_0. With Lemmas 2.2, 2.3, and 2.4 in mind, we need to track the highest-order term for each power of q. We have

$$\begin{aligned} \frac{n\,\Gamma(n)}{q\,\Gamma\left(n+1+\frac{q}{n}\right)} =& \frac{n}{q} n^{-1-q/n}\left(1+O\left(\frac{1}{n^4}\right)-\frac{q}{2n^2}+q\,O\left(\frac{1}{n^3}\right)+\sum_{k=2}^{\infty} q^k\,O\left(\frac{1}{n^{k+1}}\right)\right), \\ & \text{[1, p. 257, Expression 6.1.47]} \\ =& \frac{e^{-q/n \log n}}{q}\left(1+O\left(\frac{1}{n^4}\right)-\frac{q}{2n^2}+q\,O\left(\frac{1}{n^3}\right)+\sum_{k=2}^{\infty} q^k\,O\left(\frac{1}{n^{k+1}}\right)\right) \\ =& \frac{1}{q}\left(1-\frac{q\log n}{n}+\frac{q^2(\log n)^2}{2n^2}+\sum_{k=3}^{\infty}(-1)^k\frac{q^k(\log n)^k}{k!\,n^k}\right)\times \\ & \left(1+O\left(\frac{1}{n^4}\right)-\frac{q}{2n^2}+q\,O\left(\frac{1}{n^3}\right)+\sum_{k=2}^{\infty} q^k\,O\left(\frac{1}{n^{k+1}}\right)\right) \\ =& \frac{1}{q}+\frac{1}{q}O\left(\frac{1}{n^4}\right)-\frac{\log n}{n}+O\left(\frac{1}{n^2}\right)+\frac{q(\log n)^2}{2n^2}+q\,O\left(\frac{\log n}{n^3}\right) \\ & +\sum_{k=2}^{\infty} q^k\left(\frac{(-1)^{k+1}(\log n)^{k+1}}{(k+1)!\,n^{k+1}}+O\left(\frac{(\log n)^k}{n^{k+2}}\right)\right). \end{aligned}$$

In addition,

$$\Gamma\left(1+\frac{q}{n}\right) = 1-\frac{\gamma q}{n}+\frac{(\zeta(2)+\gamma^2)q^2}{2n^2}+\sum_{k=3}^{\infty} q^k\,O\left(\frac{1}{n^k}\right). \text{ [11, p. 935, Expression 8.321]}$$

Thus we have

$$\begin{aligned} S_0 =& \frac{1}{q}+\frac{1}{q}O\left(\frac{1}{n^4}\right)-\frac{\log n}{n}-\frac{\gamma}{n}+O\left(\frac{1}{n^2}\right)+\frac{q(\log n)^2}{2n^2}+\frac{\gamma q\log n}{n^2}+\frac{(\zeta(2)+\gamma^2)q}{2n^2} \\ & +q\,O\left(\frac{\log n}{n^3}\right)+\sum_{k=2}^{\infty} q^k\left(\frac{(-1)^{k+1}(\log n)^{k+1}}{(k+1)!\,n^{k+1}}+\frac{(-1)^{k+1}\gamma(\log n)^k}{k!\,n^{k+1}}+O\left(\frac{(\log n)^{k-1}}{n^{k+1}}\right)\right). \end{aligned}$$

Therefore, by Lemmas 2.2, 2.3, and 2.4, and the fact that $\left\{ n \atop n \right\} = 1$ and $\left\{ p \atop n \right\} = 0$ for $p < n$,

$$\sum_{q=1}^{p}(-1)^{q-1}\binom{p-1}{q-1}S_0 = \begin{cases} 1-\frac{\log n}{n}-\frac{\gamma}{n}+O\left(\frac{(\log n)^2}{n^2}\right), & p=1; \\ \frac{1}{2}-\frac{(\log n)^2}{2n^2}-\frac{\gamma\log n}{n^2}-\frac{\zeta(2)+\gamma^2}{2n^2}+O\left(\frac{(\log n)^3}{n^3}\right), & p=2; \\ \frac{1}{p}-\frac{(\log n)^p}{p\,n^p}-\frac{\gamma(\log n)^{p-1}}{n^p}+O\left(\frac{(\log n)^{p-2}}{n^p}\right), & p\geq 3. \end{cases} \quad (5)$$

We now consider, in parts, our expression (4) for S_i when $i \geq 1$ and $i = o(n^{1/2})$. (We require $i = o(n^{1/2})$ so that $\frac{i^2+q}{n-i} = o(1)$.) First, we have

$$\frac{\Gamma\left(i+\frac{i^2+q}{n-i}\right)}{i!} = \frac{1}{i}\left(\prod_{k=1}^{i-1}\frac{k+\frac{i^2+q}{n-i}}{k}\right)\Gamma\left(1+\frac{i^2+q}{n-i}\right).$$

Now (see [11, p. 935, Expression 8.321]),

$$\Gamma\left(1+\frac{i^2+q}{n-i}\right)=1-\frac{\gamma(i^2+q)}{n-i}+O\left(\frac{i^4}{n^2}\right)+q\,O\left(\frac{i^2}{n^2}\right)+\sum_{k=2}^{\infty}q^kO\left(\frac{1}{n^k}\right)$$

In addition,

$$\begin{aligned}\prod_{k=1}^{i-1}\frac{k+\frac{i^2+q}{n-i}}{k}&=\prod_{k=1}^{i-1}\left(1+\frac{i^2+q}{k(n-i)}\right)\\ &=1+\left(\frac{i^2+q}{n-i}\right)H_{i-1}+O\left(\frac{i^4(\log i)^2}{n^2}\right)+q\,O\left(\frac{i^2(\log i)^2}{n^2}\right)\\ &\quad+\sum_{k=2}^{i-1}q^k\left(\frac{1}{k!}\frac{(\log i)^k}{n^k}+\frac{\gamma}{(k-1)!}\frac{(\log i)^{k-1}}{n^k}+O\left(\frac{(\log i)^{k-2}}{n^k}\right)\right),\end{aligned}$$

by Lemma 2.10.

Therefore, by Lemma 2.9 we have

$$\begin{aligned}\frac{\Gamma\left(i+\frac{i^2+q}{n-i}\right)}{i!}&=\frac{1}{i}+\frac{i^2+q}{i(n-i)}\log i+O\left(\frac{i^3(\log i)^2}{n^2}\right)+q\,O\left(\frac{i(\log i)^2}{n^2}\right)\\ &\quad+\sum_{k=2}^{i-1}q^k\left(\frac{1}{k!}\frac{(\log i)^k}{in^k}+O\left(\frac{(\log i)^{k-2}}{in^k}\right)\right)\\ &\quad+q^i\left(\frac{-\gamma}{i!}\frac{(\log i)^{i-1}}{n^i}+O\left(\frac{(\log i)^{i-2}}{i!\,n^i}\right)\right)\\ &\quad+\sum_{k=i+1}^{\infty}q^kO\left(\frac{(\log i)^{k-2}}{in^k}\right)\\ &=\frac{1}{i}+\frac{i\log i}{n}+O\left(\frac{i^3(\log i)^2}{n^2}\right)+q\left(\frac{\log i}{in}+O\left(\frac{i(\log i)^2}{n^2}\right)\right)\\ &\quad+\sum_{k=2}^{i-1}q^k\left(\frac{1}{k!}\frac{(\log i)^k}{in^k}+O\left(\frac{(\log i)^{k-2}}{in^k}\right)\right)\\ &\quad+q^i\left(\frac{-\gamma}{i!}\frac{(\log i)^{i-1}}{n^i}+O\left(\frac{(\log i)^{i-2}}{i!\,n^i}\right)\right)\\ &\quad+\sum_{k=i+1}^{\infty}q^kO\left(\frac{(\log i)^{k-2}}{in^k}\right).\end{aligned}\tag{6}$$

Taking the rest of Expression (4), we have

$$\frac{\Gamma(n+1)\,n!}{(n^2+q)\Gamma\left(n+i+\frac{i^2+q}{n-i}\right)(n-i)!}=\frac{\Gamma(n+1)\prod_{k=n-i+1}^{n}k}{(n^2+q)\Gamma\left(n+1+\frac{i^2+q}{n-i}\right)\prod_{k=n+1}^{n+i-1}\left(k+\frac{i^2+q}{n-i}\right)}.$$

The ratio of gamma functions is (see, for example, [1, p. 257, Expression 6.1.47])

$$\begin{aligned}&\frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{i^2+q}{n-i}\right)}\\ &=n^{-(i^2+q)/(n-i)}\left(1-\frac{i^2}{2n^2}+O\left(\frac{i^4}{n^3}\right)+q\left(-\frac{1}{2n^2}+O\left(\frac{i^2}{n^3}\right)\right)+\sum_{k=2}^{\infty}q^k\,O\left(\frac{1}{n^{k+1}}\right)\right),\end{aligned}$$

and

$$\begin{aligned}n^{-(i^2+q)/(n-i)}&=\exp\left(-\frac{i^2+q}{n-i}\log n\right)\\ &=1-\frac{i^2\log n}{n}+O\left(\frac{i^4(\log n)^2}{n^2}\right)+q\left(-\frac{\log n}{n}+O\left(\frac{i^2(\log n)^2}{n^2}\right)\right)\end{aligned}$$

$$+\sum_{k=2}^{\infty} q^k \left(\frac{(-1)^k}{k!} \frac{(\log n)^k}{n^k} + O\left(\frac{i^2 (\log n)^{k+1}}{n^{k+1}} \right) \right).$$

Therefore,

$$\frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{i^2+q}{n-i}\right)} = 1 - \frac{i^2 \log n}{n} + O\left(\frac{i^4 (\log n)^2}{n^2}\right) + q\left(-\frac{\log n}{n} + O\left(\frac{i^2(\log n)^2}{n^2}\right)\right)$$
$$+\sum_{k=2}^{\infty} q^k \left(\frac{(-1)^k}{k!} \frac{(\log n)^k}{n^k} + O\left(\frac{i^2 (\log n)^{k+1}}{n^{k+1}} \right) \right). \tag{7}$$

We also have

$$\frac{\prod_{k=n-i+1}^{n} k}{(n^2+q)\prod_{k=n+1}^{n+i-1}\left(k+\frac{i^2+q}{n-i}\right)} = \frac{n}{n^2+q}\prod_{j=1}^{i-1}\frac{n+j-i}{n+j+\frac{i^2+q}{n-i}} = \frac{n}{n^2+q}\prod_{j=1}^{i-1}\frac{1-\frac{i}{n+j}}{1+\frac{i^2+q}{(n+j)(n-i)}}.$$

Now,

$$\begin{aligned}
\frac{i^2+q}{(n+j)(n-i)} &= \frac{i^2+q}{n^2}\left(\frac{1}{1+\frac{j}{n}}\right)\left(\frac{1}{1-\frac{i}{n}}\right) \\
&= \frac{i^2+q}{n^2}\left(1-\frac{j}{n}+O\left(\frac{j^2}{n^2}\right)\right)\left(1+\frac{i}{n}+O\left(\frac{i^2}{n^2}\right)\right) \\
&= \frac{i^2}{n^2} + O\left(\frac{i^3}{n^3}\right) + qO\left(\frac{1}{n^2}\right).
\end{aligned}$$

Thus

$$\frac{1}{1+\frac{i^2+q}{(n+j)(n-i)}} = 1 - \frac{i^2}{n^2} + O\left(\frac{i^3}{n^3}\right) + \sum_{k=1}^{\infty} q^k O\left(\frac{1}{n^{2k}}\right).$$

In addition,

$$1-\frac{i}{n+j} = 1 - \frac{\frac{i}{n}}{1+\frac{j}{n}} = 1 - \frac{i}{n}\left(1-\frac{j}{n}+O\left(\frac{j^2}{n^2}\right)\right) = 1-\frac{i}{n}+O\left(\frac{i^2}{n^2}\right).$$

Therefore,

$$\frac{1-\frac{i}{n+j}}{1+\frac{i^2+q}{(n+j)(n-i)}} = 1-\frac{i}{n}+O\left(\frac{i^2}{n^2}\right)+\sum_{k=1}^{\infty} q^k O\left(\frac{1}{n^{2k}}\right).$$

This implies

$$\prod_{j=1}^{i-1}\frac{1-\frac{i}{n+j}}{1+\frac{i^2+q}{(n+j)(n-i)}} = 1-\frac{i(i-1)}{n}+O\left(\frac{i^4}{n^2}\right)+\sum_{k=1}^{\infty} q^k O\left(\frac{i^k}{n^{2k}}\right).$$

Then, since

$$\frac{n}{n^2+q} = \frac{1}{n}\left(\frac{1}{1+\frac{q}{n^2}}\right) = \frac{1}{n}\left(1+\sum_{k=1}^{\infty} q^k \frac{(-1)^k}{n^{2k}}\right),$$

we have

$$\frac{n}{n^2+q}\prod_{j=1}^{i-1}\frac{1-\frac{i}{n+j}}{1+\frac{i^2+q}{(n+j)(n-i)}} = \frac{1}{n}\left(1-\frac{i(i-1)}{n}+O\left(\frac{i^4}{n^2}\right)+\sum_{k=1}^{\infty} q^k O\left(\frac{i^k}{n^{2k}}\right)\right). \tag{8}$$

Then, putting Expressions (6), (7), and (8) together, we have, for $i \geq 1$,

$$S_i = \frac{1}{in} + \frac{i\log i}{n^2} - \frac{i\log n}{n^2} - \frac{i-1}{n^2} + O\left(\frac{i^3(\log n)^2}{n^3}\right) + q\left(\frac{\log i}{in^2} - \frac{\log n}{in^2} + O\left(\frac{i(\log n)^2}{n^3}\right)\right)$$
$$+\sum_{k=2}^{i-1} q^k\left[\frac{(-1)^k(\log n)^k}{k!\, in^{k+1}} + \sum_{j=1}^{k}\frac{(-1)^{k-j}(\log i)^j(\log n)^{k-j}}{j!(k-j)!\, in^{k+1}} + O\left(\frac{(\log n)^{k-2}}{in^{k+1}}\right)\right]$$

$$+\sum_{\substack{k=i,\\ i>1}}^{\infty} q^k \Bigg[\frac{(-1)^k (\log n)^k}{k!\, i n^{k+1}} + \sum_{j=1}^{i-1} \frac{(-1)^{k-j} (\log i)^j (\log n)^{k-j}}{j!(k-j)!\, i n^{k+1}} + \frac{(-1)^{k-i+1}\gamma(\log i)^{i-1}(\log n)^{k-i}}{i!(k-i)!\, n^{k+1}} + O\left(\frac{(\log n)^{k-2}}{i n^{k+1}}\right)\Bigg].$$

Therefore, by Lemmas 2.12, 2.13, and 2.14, and the fact that $\sum_{i=1}^{\infty}(-1)^i \frac{(\log i)^k}{i}$ converges (e.g., by the alternating series test), we have, for any fixed $\alpha < 1/2$,

$$\sum_{i=1}^{n^\alpha}(-1)^i S_i = -\frac{\log 2}{n} + O\left(\frac{1}{n^{1+\alpha}}\right) + O\left(\frac{\log n}{n^{2-\alpha}}\right) + O\left(\frac{(\log n)^2}{n^{3-4\alpha}}\right) + q\left(-\frac{(\log 2)^2}{2n^2} + \frac{\gamma \log 2}{n^2} + O\left(\frac{\log n}{n^{2+\alpha}}\right) + \frac{\log 2 \log n}{n^2} + O\left(\frac{\log n}{n^{2+\alpha}}\right) + O\left(\frac{(\log n)^2}{n^{3-2\alpha}}\right)\right) + \sum_{k=2}^{\infty} q^k \left(\frac{(-1)^{k+1}\log 2 (\log n)^k}{k!\, n^{k+1}} + O\left(\frac{(\log n)^{k-1}}{n^{k+1}}\right)\right).$$

We then have, taking $\alpha = 2/5$ for $p = 1$ and $\alpha = 1/3$ for $p = 2$, together with Lemmas 2.2 and 2.3 and our result (5) for S_0,

$$\sum_{q=1}^{p}(-1)^{q-1}\binom{p-1}{q-1}\sum_{i=0}^{n^\alpha}(-1)^i S_i = \begin{cases} 1 - \frac{\log n}{n} - \frac{\log 2 + \gamma}{n} + O\left(\frac{(\log n)^2}{n^{7/5}}\right), & p = 1;\\ \frac{1}{2} - \frac{(\log n)^2}{2n^2} - \frac{(\log 2+\gamma)\log n}{n^2} - \frac{\zeta(2)+\gamma^2+2\gamma\log 2 - (\log 2)^2}{2n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & p = 2;\\ \frac{1}{p} - \frac{(\log n)^p}{p\, n^p} - \frac{(\log 2+\gamma)(\log n)^{p-1}}{n^p} + O\left(\frac{(\log n)^{p-2}}{n^p}\right), & p \geq 3. \end{cases}$$

Multiplying this expression by $-p(n^2+p)^{\underline{p}}$ and adding the result to Expression (3), we have

$$p!\sum_{y=0}^{p-1}\frac{\binom{p-1}{y}}{(p-y)!}E\left[\tau^{\underline{p-y}}(\delta(B)\geq 1;\tilde{B})\right] = \begin{cases} n\log n + (\log 2+\gamma)n + O\left(n^{3/5}(\log n)^2\right), & p = 1;\\ n^2(\log n)^2 + 2(\log 2+\gamma)n^2\log n + \left(\zeta(2)+\gamma^2+2\gamma\log 2 - (\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right), & p = 2;\\ n^p(\log n)^p + p(\log 2+\gamma)n^p(\log n)^{p-1} + O\left(n^p(\log n)^{p-2}\right), & p \geq 3. \end{cases}$$

As $\tau^p(\delta(B) \geq 1; \tilde{B})$ is the highest-powered term in $p!\sum_{y=0}^{p-1}\frac{\binom{p-1}{y}}{(p-y)!}\tau^{\underline{p-y}}(\delta(B)\geq 1;\tilde{B})$, the next highest-powered term is of degree $p-1$, and the Big-Oh error term for $E\left[\tau^{\underline{p}}(\delta(B)\geq 1;\tilde{B})\right]$ is larger than the largest term for $E\left[\tau^{\underline{p-1}}(\delta(B)\geq 1;\tilde{B})\right]$, the theorem follows. □

A few additional comments:

(i) Theorem 3.1 implies

$$\begin{aligned} &Var\left(\tau(\delta(B)\geq 1;\tilde{B})\right)\\ &= n^2(\log n)^2 + 2(\log 2+\gamma)n^2\log n + \left(\zeta(2)+\gamma^2+2\gamma\log 2-(\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right)\\ &\quad - n^2(\log n)^2 - 2(\log 2+\gamma)n^2\log n - (\log 2+\gamma)^2 n^2 + O\left(n^{8/5}(\log n)^3\right)\\ &= \left(\zeta(2) - 2(\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right). \end{aligned}$$

(ii) Numerical work indicates that, for small n ($n \leq 1500$), $(n^2+1)\left(1 - nB\left(n, 1+\frac{1}{n}\right)\right) + (\log 2)n - \log n$ (the estimate using the exact value for S_0 and the dominant term for $\sum(-1)^i S_i$ less $\log n$) has error less than 1 when used to approximate $E[\tau(\delta(B)\geq 1;\tilde{B})]$.

(iii) There is another expression for $f(n,q)$ that is perhaps of interest. We have

$$\begin{aligned}
f(n,q) &= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j}\frac{1}{n^2+q-ij} \\
&= \frac{1}{n^2+q}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j}\frac{1}{1-\frac{ij}{n^2+q}} \\
&= \frac{1}{n^2+q}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j}\sum_{k=0}^{\infty}\left(\frac{ij}{n^2+q}\right)^k \quad (\text{as } ij < n^2+q) \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{n^2+q}\right)^{k+1}\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^k\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^k \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{n^2+q}\right)^{k+1}\left((-1)^n\left\{{k \atop n}\right\}n!\right)\left((-1)^n\left\{{k \atop n}\right\}n!\right), \text{ by Lemma 2.3} \\
&= n!^2\sum_{k=n}^{\infty}\left(\frac{1}{n^2+q}\right)^{k+1}\left\{{k \atop n}\right\}^2.
\end{aligned}$$

The convergence of this infinite sum is fairly slow, however, and thus is not as helpful as the alternating sum we use in obtaining an asymptotic expression for $f(n,q)$.

4. Moments for time to first matching. We now determine asymptotic expressions for the moments of $\tau(\text{match}; \tilde{B})$. These turn out to be the same as those for $\tau(\delta(B) \geq 1; \tilde{B})$ obtained in Section 3.

Theorem 4.1

$$E\left[\tau^p(match; \tilde{B})\right] = \begin{cases} n\log n + (\log 2 + \gamma)n + O\left(n^{3/5}(\log n)^2\right), & p = 1; \\ n^2(\log n)^2 + 2(\log 2+\gamma)n^2\log n & \\ \quad + \left(\zeta(2) + \gamma^2 + 2\gamma\log 2 - (\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right), & p = 2; \\ n^p(\log n)^p + p(\log 2+\gamma)n^p(\log n)^{p-1} + O\left(n^p(\log n)^{p-2}\right), & p \geq 3. \end{cases}$$

Proof. As in the proof of Theorem 3.1, let $B_{n,m}$ be a graph chosen at random from the set of all bipartite graphs on V_1 and V_2 with $|V_1| = |V_2| = n$ that contain exactly m edges. Lemma 2.1 implies

$$\begin{aligned}
E[\tau^{\underline{p}}(\text{match}; \tilde{B})] &= \sum_{m=0}^{n^2} pm^{\underline{p-1}}P(\tau(\text{match}; \tilde{B}) > m) \\
&= \sum_{m=0}^{n^2} pm^{\underline{p-1}}P(B_{n,m} \text{ has no matching}) \\
&= \sum_{m=0}^{n^2} pm^{\underline{p-1}}P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) = 0) \\
&\quad + \sum_{m=0}^{n^2} pm^{\underline{p-1}}P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) \\
&= \sum_{m=0}^{n^2} pm^{\underline{p-1}}P(\delta(B_{n,m}) = 0) \\
&\quad + \sum_{m=0}^{n^2} pm^{\underline{p-1}}P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1).
\end{aligned}$$

Theorem 3.1 gives us an expression for the first sum, which is equal to $E\left[\tau^p(\delta(B) \geq 1; \tilde{B})\right]$. As we shall see, this sum dominates the second sum asymptotically. We split the second sum into three pieces. As m increases from 0 to n^2, the event $\delta(B_{n,m}) \geq 1$ becomes much more likely once $m \approx \frac{1}{2}n\log n$ [3, p. 77], and the event that $B_{n,m}$ has no matching becomes much less likely once $m \approx n \log n$ [8]. Splitting the sum near these two places thus turns out to be helpful.

Claim 1. For $0 \leq k \leq 1$, $\sum_{m=0}^{\lfloor n(1-k)\log n \rfloor} pm^{\underline{p-1}} P(B_{n,m}$ has no matching and $\delta(B_{n,m}) \geq 1) = O(n^{p-2k}(\log n)^{p-1})$.

Proof. If $v \in V_1$ then, for fixed m, $P(\delta(v) = 0)$ is the probability that all m of the actual edges in $B_{n,m}$ are distributed among the $n^2 - n$ potential edges that are not incident on v. Thus we have

$$\begin{aligned} P(\delta(v)=0) &= \frac{\binom{n^2-n}{m}}{\binom{n^2}{m}} = \frac{(n^2-n)!(n^2-m)!}{n^2!(n^2-n-m)!} \\ &= \frac{(n^2-m)(n^2-m-1)\cdots(n^2-m-n+1)}{n^2(n^2-1)\cdots(n^2-n+1)} \\ &= \left(1-\frac{m}{n^2}\right)\left(1-\frac{m}{n^2-1}\right)\cdots\left(1-\frac{m}{n^2-n+1}\right) \\ &\geq \left(1-\frac{m}{n^2-n+1}\right)^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) \\ &\leq P(\delta(B_{n,m}) \geq 1) \\ &\leq P(\delta(v_1) \geq 1, \delta(v_2) \geq 1, \ldots, \delta(v_n) \geq 1), \text{ where } V_1 = \{v_1, v_2, \ldots, v_n\} \\ &= P(\delta(v_1) \geq 1) P(\delta(v_2) \geq 1 | \delta(v_1) \geq 1) \cdots P(\delta(v_n) \geq 1 | \delta(v_1) \geq 1, \delta(v_2) \geq 1, \ldots, \delta(v_{n-1}) \geq 1) \\ &\leq P(\delta(v_1) \geq 1) P(\delta(v_2) \geq 1) \cdots P(\delta(v_n) \geq 1) \\ &\leq \left(1-\left(1-\frac{m}{n^2-n+1}\right)^n\right)^n. \end{aligned}$$

Let $m = n\log n - cn$, $0 \leq c \leq \log n$. Then

$$P(\delta(B_{n,m}) \geq 1) \leq \left(1-\left(1-\frac{n\log n - cn}{n^2-n+1}\right)^n\right)^n.$$

Of course, the dominant term in $n^2 - n + 1$ is n^2, and we know that, as $n \to \infty$,

$$\frac{(1-\frac{n\log n - cn}{n^2})^n}{e^{-\log n + c}} \to 1.$$

For the range of values of n and c we consider, it is not too hard to show that $(1 - \frac{n\log n - cn}{n^2-n+1})^n$ is never very different from $e^{-\log n + c}$, and, in fact, for $0 \leq c \leq \log n$, $n \geq 1$,

$$\left(1-\frac{n\log n - cn}{n^2-n+1}\right)^n \geq 0.43 e^{-\log n + c} = \frac{0.43e^c}{n},$$

with the minimum value of

$$\frac{(1-\frac{n\log n - cn}{n^2-n+1})^n}{e^{-\log n + c}}$$

occurring when $c = 0$ and $n = 4$. Therefore, if $m = n\log n - cn$, then

$$P(\delta(B_{n,m}) \geq 1) \leq \left(1-\frac{0.43e^c}{n}\right)^n \leq e^{-0.43e^c}.$$

Thus we have

$$\begin{aligned} &\sum_{m=0}^{\lfloor n(1-k)\log n \rfloor} pm^{\underline{p-1}} P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) \\ &\leq p(n\log n)^{\underline{p-1}} \sum_{m=0}^{\lfloor n(1-k)\log n \rfloor} P(\delta(B_{n,m}) \geq 1) \end{aligned}$$

$$
\begin{aligned}
&\le p(n\log n)^{\underline{p-1}} \sum_{c=\lfloor k\log n\rfloor}^{\lceil \log n\rceil} \sum_{m=\lfloor n\log n\rfloor-(c+1)n+1}^{\lfloor n\log n\rfloor - cn} P(\delta(B_{n,m})\ge 1)\\
&\le p(n\log n)^{\underline{p-1}} \sum_{c=\lfloor k\log n\rfloor}^{\lceil \log n\rceil} \sum_{m=\lfloor n\log n\rfloor-(c+1)n+1}^{\lfloor n\log n\rfloor - cn} P(\delta(B_{n,\lfloor n\log n\rfloor - cn})\ge 1)\\
&\le p(n\log n)^{\underline{p-1}} \sum_{c=\lfloor k\log n\rfloor}^{\lceil \log n\rceil} \sum_{m=\lfloor n\log n\rfloor-(c+1)n+1}^{\lfloor n\log n\rfloor - cn} e^{-0.43e^c}\\
&= pn(n\log n)^{\underline{p-1}} \sum_{c=\lfloor k\log n\rfloor}^{\lceil \log n\rceil} e^{-0.43e^c}\\
&\le pn(n\log n)^{\underline{p-1}} \sum_{c=\lfloor k\log n\rfloor}^{\infty} e^{-0.43e^c}\\
&\le pn(n\log n)^{\underline{p-1}} \sum_{j=-1}^{\infty} e^{-0.43n^k e^j}\\
&\le pn(n\log n)^{\underline{p-1}} \left(e^{-(0.43/e)n^k} + \sum_{j=1}^{\infty} e^{-0.43n^k j}\right), \text{ as } e^j \ge j+1\\
&= pn(n\log n)^{\underline{p-1}} \left(e^{-(0.43/e)n^k} + \frac{e^{-0.43n^k}}{1-e^{-0.43n^k}}\right)\\
&< pn(n\log n)^{\underline{p-1}} \left(\frac{e^2}{(0.43)^2 n^{2k}} + \frac{1}{(0.43)^2 n^{2k}-1}\right), \text{ as } e^{-x} < \frac{1}{x^2} \text{ for } x>0\\
&= O(n^{p-2k}(\log n)^{p-1}).
\end{aligned}
$$

□

CLAIM 2. For $0 \le k < \frac{1}{2}$, $\sum_{m=\lceil n(1-k)\log n\rceil}^{\lfloor n\log n\rfloor} pm^{\underline{p-1}} P(B_{n,m}$ has no matching and $\delta(B_{n,m}) \ge 1) = O\left(n^{p+2k-1}(\log n)^{p-1}\right)$.

PROOF. Suppose $m = \frac{n}{2}\log n + cn$, where $0 < c \le \frac{1}{2}\log n$. Frieze [9] shows that

$$P(B_{n,m} \text{ has no matching} | \delta(B_{n,m}) \ge 1) \le O\left(e^{-2c} + n^{-1/2-3c/\log n}\right).$$

Thus

$$P(B_{n,m} \text{ has no matching} | \delta(B_{n,m}) \ge 1) = O\left(e^{-2c} + n^{-1/2}e^{-3c}\right) = O(e^{-2c}),$$

as $n^{1/\log n} = e$ and $c > 0$.

Since

$$
\begin{aligned}
&P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \ge 1)\\
&\quad = P(B_{n,m} \text{ has no matching} | \delta(B_{n,m}) \ge 1) P(\delta(B_{n,m}) \ge 1)\\
&\quad \le P(B_{n,m} \text{ has no matching} | \delta(B_{n,m}) \ge 1),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{m=\lceil n(1-k)\log n\rceil}^{\lfloor n\log n\rfloor} pm^{\underline{p-1}} P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \ge 1)\\
&\le p(n\log n)^{\underline{p-1}} \sum_{m=\lceil n(1-k)\log n\rceil}^{\lfloor n\log n\rfloor} P(B_{n,m} \text{ has no matching} | \delta(B_{n,m}) \ge 1)\\
&\le p(n\log n)^{\underline{p-1}} \sum_{c=\lfloor(\frac{1}{2}-k)\log n\rfloor}^{\lceil \frac{1}{2}\log n\rceil} \sum_{m=\lfloor \frac{n}{2}\log n\rfloor + cn}^{\lfloor \frac{n}{2}\log n\rfloor + (c+1)n-1} P(B_{n,m} \text{ has no matching} | \delta(B_{n,m}) \ge 1)
\end{aligned}
$$

$$
\begin{aligned}
&\leq p(n\log n)^{\underline{p-1}} \sum_{c=\lfloor(\frac{1}{2}-k)\log n\rfloor}^{\lceil \frac{1}{2}\log n\rceil} \sum_{m=\lfloor \frac{n}{2}\log n\rfloor + cn}^{\lfloor \frac{n}{2}\log n\rfloor+(c+1)n-1} O(e^{-2c}) \\
&= pn(n\log n)^{\underline{p-1}} \sum_{c=\lfloor(\frac{1}{2}-k)\log n\rfloor}^{\lceil \frac{1}{2}\log n\rceil} O(e^{-2c}) \\
&\leq pn(n\log n)^{\underline{p-1}} \sum_{c=\lfloor(\frac{1}{2}-k)\log n\rfloor}^{\lceil \frac{1}{2}\log n\rceil} Ke^{-2c}, \text{ for some constant } K \\
&\leq pn(n\log n)^{\underline{p-1}} \sum_{c=\lfloor(\frac{1}{2}-k)\log n\rfloor}^{\infty} Ke^{-2c} \\
&\leq Kpn(n\log n)^{\underline{p-1}} \left(e^{(-1+2k)\log n} \sum_{j=-1}^{\infty} e^{-2j} \right) \\
&= n(n\log n)^{\underline{p-1}} n^{2k-1} O(1) \\
&= O\left(n^{p+2k-1}(\log n)^{p-1}\right).
\end{aligned}
$$

□

Claim 3. $\sum_{m=\lceil n\log n\rceil}^{n^2} pm^{\underline{p-1}} P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) = O\left(n^{p-1/2}(\log n)^{p+1}\right).$

Proof. Erdős and Rényi [8] prove that, for $m = n\log n + cn + o(n)$,

$$P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) \leq \frac{A(\log n)^2}{\sqrt{n} - A(\log n)^2}, \tag{9}$$

where A is a positive constant depending only on c. Holding A fixed and summing this result as m ranges from $n\log n$ to n^2 yields an expression on the order of $n\sqrt{n}(\log n)^2$, which is larger than we want. However, we still use the approach of Erdős and Rényi: The dependence of A upon c, which they do not need for their result, turns out to be crucial for ours. We also fill in some details not provided in their paper.

As with Erdős and Rényi, define $Q_k(n,m)$ to be the probability that there can be found k rows and $n-k-1$ columns or k columns and $n-k-1$ rows that contain all the ones in the adjacency matrix representation of $B_{n,m}$, and k is the least number with this property. Then, by the theorem of Frobenius and Kőnig (see, for example, Minc and Marcus [14, p. 31]), $P(B_{n,m}$ has no matching and $\delta(B_{n,m}) \geq 1) = \sum_{k=1}^{\lfloor \frac{n-1}{2}\rfloor} Q_k(n,m)$.

Erdős and Rényi show that

$$Q_k(n,m) \leq 2\binom{n}{k}\binom{n}{k+1}\binom{k+1}{2}^k \frac{\binom{n(n-k-1)+k(k-1)}{m-2k}}{\binom{n^2}{m}}. \tag{10}$$

However, two lines later they arrive at Expression (9). We need the intermediate steps in this calculation.

First, we upper-bound the binomial expression in Expression (10).

$$
\begin{aligned}
&\frac{\binom{n(n-k-1)+k(k-1)}{m-2k}}{\binom{n^2}{m}} \\
&= \frac{(n^2-kn-n+k^2-k)!\, m!\, (n^2-m)!}{(m-2k)!\, (n^2-kn-n+k^2+k-m)!\, n^2!} \\
&= \frac{(n^2-m)(n^2-m-1)\cdots(n^2-m-kn-n+k^2+k+1)}{n^2(n^2-1)\cdots(n^2-kn-n+k^2+k+1)} \times \\
&\quad \frac{m(m-1)\cdots(m-2k+1)}{(n^2-kn-n+k^2+k)\cdots(n^2-kn-n+k^2-k+1)} \\
&= \left(1-\frac{m}{n^2}\right)\left(1-\frac{m}{n^2-1}\right)\cdots\left(1-\frac{m}{n^2-kn-n+k^2+k+1}\right) \times
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{m}{n^2-kn-n+k^2+k}\right)\cdots\left(\frac{m-2k+1}{n^2-kn-n+k^2-k+1}\right)\\
&\le\left(1-\frac{m}{n^2}\right)^{kn+n-k^2-k}\left(\frac{m}{n^2-kn-n+k^2+k}\right)^{2k}.
\end{aligned}
$$

Now, $n^2-kn-n+k^2+k$ is minimized when $k=\frac{n-1}{2}$. The minimum value is $3n^2/4-n/2-1/4$, which, for $n\ge 36$, is larger than $2n^2/e$. Thus, for sufficiently large n we have

$$
\begin{aligned}
\frac{\binom{n(n-k-1)+k(k-1)}{m-2k}}{\binom{n^2}{m}} &\le \left(1-\frac{m}{n^2}\right)^{kn+n-k^2-k}\left(\frac{em}{2n^2}\right)^{2k}\\
&=\left(\frac{e}{2}\right)^{2k}\left(1-\frac{m}{n^2}\right)^{(k+1)(n-k)}\left(\frac{m}{n^2}\right)^{2k}. \qquad (11)
\end{aligned}
$$

We now show that if $m=n\log n+cn+r$, $0\le c\le n-\log n$, $0\le r<n$, $1\le k\le\lfloor\frac{n-1}{2}\rfloor$,

$$
Q_k(n,m)\le e^{1-c/2}\left(\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n+c+\frac{r}{n}\right)^2\right)^k.
$$

Case 1. $k\le\frac{n}{2}-1$.

For this case, $(k+1)(n-k)\ge k(\frac{n}{2}+1)+n-k=n(\frac{k}{2}+1)$. Then, since $\binom{n}{k}\le(\frac{ne}{k})^k$, by Expressions (10) and (11) we have

$$
\begin{aligned}
Q_k(n,m)&\le 2\left(\frac{ne}{k}\right)^k\left(\frac{ne}{k+1}\right)^{k+1}\left(\frac{k(k+1)}{2}\right)^k\left(\frac{e}{2}\right)^{2k}\left(\left(1-\frac{m}{n^2}\right)^n\right)^{k/2+1}\left(\frac{m}{n^2}\right)^{2k}\\
&=\frac{e^{4k+1}n}{2^{3k-1}(k+1)}\left(\left(1-\frac{m}{n^2}\right)^n\right)^{k/2+1}\left(\frac{m}{n}\right)^{2k}\\
&\le\frac{e^{4k+1}n}{2^{3k-1}(k+1)}\left(e^{-m/n}\right)^{k/2+1}\left(\frac{m}{n}\right)^{2k}\\
&=\frac{2en}{k+1}e^{-m/n}\left(\frac{e^4m^2}{8n^2}e^{-m/(2n)}\right)^k.
\end{aligned}
$$

For $m=n\log n+cn+r$, we have

$$
\begin{aligned}
Q_k(n,m)&\le\frac{2en}{k+1}\left(e^{-\log n-c-r/n}\right)\left(\frac{e^4}{8}\left(\log n+c+\frac{r}{n}\right)^2\left(e^{-\log n-c-r/n}\right)^{1/2}\right)^k\\
&=\frac{2}{k+1}e^{1-c-r/n}\left(\frac{1}{8\sqrt{n}}e^{4-c/2-r/(2n)}\left(\log n+c+\frac{r}{n}\right)^2\right)^k\\
&\le e^{1-c/2}\left(\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n+c+\frac{r}{n}\right)^2\right)^k.
\end{aligned}
$$

Case 2. $k=\frac{n}{2}-\frac{1}{2}$.

Here, $k+1=\frac{n}{2}+\frac{1}{2}$, and so $\binom{n}{k+1}=\binom{n}{k}$. Also, $(k+1)(n-k)\ge(k+1)\frac{n+1}{2}\ge\frac{n}{2}(k+1)$. Thus we have, by Expressions (10) and (11),

$$
\begin{aligned}
Q_k(n,m)&\le 2\left(\frac{ne}{k}\right)^k\left(\frac{ne}{k}\right)^k\left(\frac{k(k+1)}{2}\right)^k\left(\frac{e}{2}\right)^{2k}\left(\left(1-\frac{m}{n^2}\right)^n\right)^{(k+1)/2}\left(\frac{m}{n^2}\right)^{2k}\\
&=\frac{e^{4k}}{2^{3k-1}}\left(1+\frac{1}{k}\right)^k\left(\left(1-\frac{m}{n^2}\right)^n\right)^{(k+1)/2}\left(\frac{m}{n}\right)^{2k}\\
&\le\frac{e^{4k+1}}{2^{3k-1}}\left(e^{-m/(2n)}\right)^{k+1}\left(\frac{m}{n}\right)^{2k}\\
&=2e^{1-m/(2n)}\left(\frac{e^4m^2}{8n^2}e^{-m/(2n)}\right)^k.
\end{aligned}
$$

For $m = n \log n + cn + r$, $n \geq 4$, we have

$$\begin{aligned}
Q_k(n,m) &\leq 2e\left(e^{-\log n - c - r/n}\right)^{1/2}\left(\frac{e^4}{8}\left(\log n + c + \frac{r}{n}\right)^2\left(e^{-\log n - c - r/n}\right)^{1/2}\right)^k \\
&= \frac{2}{\sqrt{n}}e^{1-c/2-r/(2n)}\left(\frac{1}{8\sqrt{n}}e^{4-c/2-r/(2n)}\left(\log n + c + \frac{r}{n}\right)^2\right)^k \\
&\leq e^{1-c/2}\left(\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2\right)^k.
\end{aligned}$$

Now, there exists N such that for any $n \geq N$ and any $c \geq 0$, $0 \leq r < n$,

$$\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2 < \frac{1}{2}.$$

Therefore, for $m = n \log n + cn + r$, $0 \leq c \leq n - \log n$, $0 \leq r < n$, and $n \geq N$ we have

$$\begin{aligned}
\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} Q_k(n,m) &\leq \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} e^{1-c/2}\left(\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2\right)^k \\
&\leq \sum_{k=1}^{\infty} e^{1-c/2}\left(\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2\right)^k \\
&= e^{1-c/2}\frac{\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2}{1 - \frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2} \\
&\leq e^{1-c/2}\frac{\frac{e^{4-c/2}}{8\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2}{1 - \frac{1}{2}} \\
&= \frac{e^{5-c}}{4\sqrt{n}}\left(\log n + c + \frac{r}{n}\right)^2.
\end{aligned}$$

Thus, for $n \geq N$,

$$\begin{aligned}
&\sum_{m=\lceil n \log n \rceil}^{n^2} pm^{\underline{p-1}} P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) \\
&\leq \sum_{c=0}^{\lceil n - \log n \rceil} \sum_{m=\lceil n \log n \rceil + cn}^{\lceil n \log n \rceil + (c+1)n - 1} pm^{\underline{p-1}} P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) \\
&= p\sum_{c=0}^{\lceil n - \log n \rceil} \sum_{r=0}^{n-1} (\lceil n \log n \rceil + cn + r)^{\underline{p-1}} \times \\
&\qquad P(B_{n,\lceil n \log n \rceil + cn + r} \text{ has no matching and } \delta(B_{n,\lceil n \log n \rceil + cn + r}) \geq 1) \\
&\leq p\sum_{c=0}^{\lceil n - \log n \rceil} \sum_{r=0}^{n-1} (n \log n + (c+1)n)^{\underline{p-1}} \frac{e^{5-c}}{4\sqrt{n}}\left(\log n + c + \frac{r+1}{n}\right)^2 \\
&\leq p\sum_{c=0}^{\infty} \sum_{r=0}^{n-1} (n \log n + (c+1)n)^{\underline{p-1}} \frac{e^{5-c}}{4\sqrt{n}}(\log n + c + 1)^2 \\
&\leq pn\sum_{c=0}^{\infty} (n \log n + (c+1)n)^{\underline{p-1}} \frac{e^{5-c}}{4\sqrt{n}}(\log n + c + 1)^2 \\
&= O\left(n^{p-1/2}(\log n)^{p+1}\right).
\end{aligned}$$

This completes the proof of Claim 3. □

Taking $k = 1/4$ in Claims 1 and 2, and combining them with Claim 3, we have

$$\sum_{m=0}^{n^2} pm^{\underline{p-1}} P(B_{n,m} \text{ has no matching and } \delta(B_{n,m}) \geq 1) = O\left(n^{p-1/2}(\log n)^{p+1}\right).$$

Combining this result with that obtained in Theorem 3.1 proves that

$$E\left[\tau^{\underline{p}}(\text{match}; \tilde{B})\right] = \begin{cases} n\log n + (\log 2 + \gamma)n + O\left(n^{3/5}(\log n)^2\right), & p = 1; \\ n^2(\log n)^2 + 2(\log 2 + \gamma)n^2 \log n \\ \quad + \left(\zeta(2) + \gamma^2 + 2\gamma\log 2 - (\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right), & p = 2; \\ n^p(\log n)^p + p(\log 2 + \gamma)n^p(\log n)^{p-1} + O\left(n^p(\log n)^{p-2}\right), & p \geq 3. \end{cases}$$

As $\tau^p(\text{match}; \tilde{B})$ is the highest-powered term in $\tau^{\underline{p}}(\text{match}; \tilde{B})$, the next highest-powered term is of degree $p - 1$, and the Big-Oh error term for $E\left[\tau^p(\text{match}; \tilde{B})\right]$ is larger than the largest term for $E\left[\tau^{p-1}(\text{match}; \tilde{B})\right]$, the theorem follows. □

Theorem 4.1 implies $Var\left(\tau(\text{match}; \tilde{B})\right) = \left(\zeta(2) - 2(\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right)$, just as with $Var\left(\tau(\delta(B) \geq 1; \tilde{B})\right)$.

5. Moments for the bottleneck assignment problem. We now apply the results obtained in Theorem 4.1 for the moments of $\tau(\text{match}; \tilde{B})$ to give a method for determining moments for certain random bottleneck assignment problems.

Let R denote the rank of the optimal cost c_n^* of an $n \times n$ bottleneck assignment problem. Then, by Lemma 1.1 and Theorem 4.1 we have

Corollary 5.1

$$E\left[R^p\right] = \begin{cases} n\log n + (\log 2 + \gamma)n + O\left(n^{3/5}(\log n)^2\right), & p = 1; \\ n^2(\log n)^2 + 2(\log 2 + \gamma)n^2 \log n \\ \quad + \left(\zeta(2) + \gamma^2 + 2\gamma\log 2 - (\log 2)^2\right)n^2 + O\left(n^{5/3}(\log n)^2\right), & p = 2; \\ n^p(\log n)^p + p(\log 2 + \gamma)n^p(\log n)^{p-1} + O\left(n^p(\log n)^{p-2}\right), & p \geq 3. \end{cases}$$

We then have our major result on the moments of the bottleneck assignment problem.

Theorem 5.1 *Let c_n^* be the optimal cost of an $n \times n$ bottleneck assignment problem whose costs are iid random variables from a continuous distribution with cdf F, and let $Q = F^{-1}$. Suppose that $Q(0) = 0$ and that Q can be expanded in a Maclaurin series. Let $m = \min_{d \geq 0}\{Q^{(d)}(0) \neq 0\}$. Then, for $p \geq 1$, we have*

$$E[(c_n^*)^p] = \begin{cases} Q'(0)\left(\frac{\log n}{n} + \frac{\log 2 + \gamma}{n}\right) + O\left(\frac{(\log n)^2}{n^{7/5}}\right), & mp = 1; \\ \left(\frac{Q^{(m)}(0)}{m!}\right)^p \left(\frac{(\log n)^2}{n^2} + \frac{2(\log 2+\gamma)\log n}{n^2} + \frac{\zeta(2)+\gamma^2+2\gamma\log 2-(\log 2)^2}{n^2}\right) + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & mp = 2; \\ \left(\frac{Q^{(m)}(0)}{m!}\right)^p \left(\frac{(\log n)^{mp}}{n^{mp}} + \frac{mp(\log 2+\gamma)(\log n)^{mp-1}}{n^{mp}}\right) + O\left(\frac{(\log n)^{mp-2}}{n^{mp}}\right), & mp \geq 3. \end{cases}$$

Proof. As mentioned in the introduction, two fundamental properties of the bottleneck assignment problem are 1) its optimal solution is taken by one of the c_{ij}'s, and 2) the optimal solution depends only on the relative rank of the c_{ij}'s and not on their numerical values [5, p. 172]. Thus $E[(c_n^*)^p] = E[E[(c_n^*)^p|R]] = E[E[X_{(R)}^p]]$, where $X_{(R)}$ is the Rth order statistic of a random sample of size n^2 from the distribution with cdf F. Since $X_{(R)}$ and $F^{-1}(U_{(R)})$ have the same distribution, where $U_{(R)}$ is the Rth order statistic from a random sample of size n^2 from a $U(0,1)$ distribution [7, p. 15], we have $E[E[X_{(R)}^p]] = E[E[(Q(U_{(R)}))^p]]$. Therefore,

$$\begin{aligned} E[(c_n^*)^p] &= E[E[(Q(U_{(R)}))^p]] \\ &= E\left[E\left[\left(\frac{Q^{(m)}(0)}{m!}U_{(R)}^m + O\left(U_{(R)}^{m+1}\right)\right)^p\right]\right] \end{aligned}$$

$$= E\left[E\left[\sum_{k=0}^{p}\binom{p}{k}\left(\frac{Q^{(m)}(0)}{m!}\right)^k U_{(R)}^{mk}\ O\left(U_{(R)}^{(m+1)(p-k)}\right)\right]\right]$$
$$= E\left[E\left[\left(\frac{Q^{(m)}(0)}{m!}\right)^p U_{(R)}^{mp} + O\left(U_{(R)}^{mp+1}\right)\right]\right].$$

Because $U_{(R)} \sim \text{beta}(R, n^2 - R + 1)$ [6, p. 233], and the kth moment of a beta(a, b) distribution is [6, p. 108]

$$\frac{\Gamma(a+k)\Gamma(a+b)}{\Gamma(a+b+k)\Gamma(a)},$$

we have

$$E[U_{(R)}^k] = \frac{\Gamma(R+k)\Gamma(n^2+1)}{\Gamma(n^2+1+k)\Gamma(R)} = \frac{(R+k-1)(R+k-2)\cdots R}{(n^2+k)(n^2+k-1)\cdots(n^2+1)} = \frac{R^k}{n^{2k}} + O\left(\frac{R^{k-1}}{n^{2k}}\right).$$

Therefore,

$$E[(c_n^*)^p] = E\left[E\left[\left(\frac{Q^{(m)}(0)}{m!}\right)^p U_{(R)}^{mp} + O\left(U_{(R)}^{mp+1}\right)\right]\right]$$
$$= E\left[\left(\frac{Q^{(m)}(0)}{m!}\right)^p \frac{R^{mp}}{n^{2mp}} + O\left(\frac{R^{mp-1}}{n^{2mp}}\right) + O\left(\frac{R^{mp+1}}{n^{2mp+2}}\right)\right].$$

Applying Corollary 5.1 completes the proof. □

If $F^{-1}(0) = a \neq 0$, but the rest of the hypotheses of Theorem 5.1 hold, then one can apply the results of Theorem 5.1 to determine $E[(c_n^* - a)^p]$ and then use this result to determine $E[(c_n^*)^p]$.

There are many continuous distributions for which F^{-1} can be expanded in a Maclaurin series. (The Gaussian, or normal, distribution is probably the most important of those that cannot.) We illustrate Theorem 5.1 by considering a few continuous distributions for which our approach can be applied.

Corollary 5.2 *Let c_n^* be the optimal solution to an $n \times n$ bottleneck assignment problem whose costs are iid random variables from a $U[0,1]$ distribution. Then*

$$E[(c_n^*)^p] = \begin{cases} \frac{\log n}{n} + \frac{\log 2 + \gamma}{n} + O\left(\frac{(\log n)^2}{n^{7/5}}\right), & p = 1; \\ \frac{(\log n)^2}{n^2} + \frac{2(\log 2+\gamma)\log n}{n^2} + \frac{\zeta(2)+\gamma^2+2\gamma\log 2-(\log 2)^2}{n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & p = 2; \\ \frac{(\log n)^p}{n^p} + \frac{p(\log 2+\gamma)(\log n)^{p-1}}{n^p} + O\left(\frac{(\log n)^{p-2}}{n^p}\right), & p \geq 3, \end{cases}$$

and

$$Var(c_n^*) = \frac{\zeta(2) - 2(\log 2)^2}{n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right).$$

Proof. For a continuous $U[0,1]$ distribution, $F(x) = x$. Thus $Q(x) = F^{-1}(x) = x$. Since $Q(0) = 0$, $Q'(0) = 1$, and $Q^{(k)}(0) = 0$ for all $k \geq 2$, the result for $E[(c_n^*)^p]$ follows from Theorem 5.1. The expression for the variance follows immediately from that for $E[(c_n^*)^p]$. □

Corollary 5.2 improves on the previous best-known bounds, due to Pferschy [15], for $E[c_n^*]$ when costs are chosen independently from the $U[0,1]$ distribution. His results are that, for $n > 78$,

$$1 - nB\left(n, 1 + \frac{1}{n}\right) \leq E[c_n^*] < 1 - \left[\frac{2}{n(n+2)}\right]^{2/n} \frac{n}{n+2} + \frac{123}{610n},$$

which implies

$$\frac{\log n + \gamma}{n} + O\left(\frac{(\log n)^2}{n^2}\right) \leq E[c_n^*] \leq \frac{4\log n}{n} + O\left(\frac{1}{n}\right).$$

Interestingly enough, Pferschy's lower bound of $1 - nB(n, 1 + 1/n)$ for $E[c_n^*]$ is exactly what one would obtain by using only S_0 to lower bound $E[\tau(\delta(B) \geq 1; \tilde{B})]$ in our proof of Theorem 3.1.

Corollary 5.3 *Let c_n^* be the optimal solution to an $n \times n$ bottleneck assignment problem whose costs are iid random variables from an exponential(λ) distribution. Then*

$$E[(c_n^*)^p] = \begin{cases} \frac{\log n}{\lambda n} + \frac{\log 2+\gamma}{\lambda n} + O\left(\frac{(\log n)^2}{n^{7/5}}\right), & p=1; \\ \frac{(\log n)^2}{\lambda^2 n^2} + \frac{2(\log 2+\gamma)\log n}{\lambda^2 n^2} + \frac{\zeta(2)+\gamma^2+2\gamma\log 2-(\log 2)^2}{\lambda^2 n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & p=2; \\ \frac{(\log n)^p}{\lambda^p n^p} + \frac{p(\log 2+\gamma)(\log n)^{p-1}}{\lambda^p n^p} + O\left(\frac{(\log n)^{p-2}}{n^p}\right), & p \geq 3, \end{cases}$$

and

$$Var(c_n^*) = \frac{\zeta(2) - 2(\log 2)^2}{\lambda^2 n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right).$$

Proof. If the probability density function is $f(x) = \lambda e^{-\lambda x}$, then $F(x) = 1 - e^{-\lambda x}$, and $Q(x) = F^{-1}(x) = -\frac{1}{\lambda}\log(1-x)$. Since $Q^{(k)}(0)$ exists and is finite for all $k \geq 0$, $Q(0) = 0$, and $Q'(0) = \frac{1}{\lambda(1-x)}|_{x=0} = \frac{1}{\lambda}$, the expression for $E[(c_n^*)^p]$ follows from Theorem 5.1. The variance follows from the expression for $E[(c_n^*)^p]$. □

The $\chi^2(1)$ distribution gives us a different kind of example, as $Q'(0) = 0$.

Corollary 5.4 *Let c_n^* be the optimal solution to an $n \times n$ bottleneck assignment problem whose costs are iid random variables from a $\chi^2(1)$ distribution. Then*

$$E[(c_n^*)^p] = \begin{cases} \frac{\pi(\log n)^2}{2n^2} + \frac{\pi(\log 2+\gamma)\log n}{n^2} + \frac{\pi\left(\zeta(2)+\gamma^2+2\gamma\log 2-(\log 2)^2\right)}{2n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & p=1; \\ \frac{\pi^p(\log n)^{2p}}{2^p n^{2p}} + \frac{p\pi^p(\log 2+\gamma)(\log n)^{2p-1}}{2^{p-1}n^{2p}} + O\left(\frac{(\log n)^{2p-2}}{n^{2p}}\right), & p \geq 2, \end{cases}$$

and

$$Var(c_n^*) = O\left(\frac{(\log n)^2}{n^4}\right).$$

Proof. For a $\chi^2(1)$ distribution,

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t/2}t^{-1/2}\,dt.$$

With the change of variable $y = \sqrt{t/2}$ we obtain

$$F(x) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x/2}} e^{-y^2}\,dy = \operatorname{erf}\left(\sqrt{x/2}\right),$$

where erf(x) is the error function. Thus $Q(x) = F^{-1}(x) = 2(\operatorname{erf}^{-1}(x))^2$. Now, $\operatorname{erf}^{-1}(x)$ has a known Maclaurin series whose first two terms are $\frac{\sqrt{\pi}}{2}x + \frac{\pi^{3/2}}{24}x^3$ [19]. Thus the first two nonzero terms in the Maclaurin expansion for $Q(x)$ are $\frac{\pi}{2}x^2$ and $\frac{\pi^2}{12}x^4$. Therefore, $Q(0) = Q'(0) = 0$, $Q''(0) = \pi$, and the result follows from Theorem 5.1. For the variance calculation the dominant terms in $E[(c_n^*)^2]$ and $(E[c_n^*])^2$ cancel. □

As a final example, we can also use Theorem 5.1 when costs are chosen from a beta(a,b) distribution, provided $1/a \in \mathbb{Z}^+$.

Corollary 5.5 *Let c_n^* be the optimal solution to an $n \times n$ bottleneck assignment problem whose costs are iid random variables from a beta(a,b) distribution, where $1/a \in \mathbb{Z}^+$. Then*

$$E[(c_n^*)^p] = \begin{cases} \frac{\log n}{bn} + \frac{\log 2+\gamma}{bn} + O\left(\frac{(\log n)^2}{n^{7/5}}\right), & a=p=1; \\ (aB(a,b))^2\left(\frac{(\log n)^2}{n^2} + \frac{2(\log 2+\gamma)\log n}{n^2} + \frac{\zeta(2)+\gamma^2+2\gamma\log 2-(\log 2)^2}{n^2}\right) + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & p/a=2; \\ (aB(a,b))^{p/a}\left(\frac{(\log n)^{p/a}}{n^{p/a}} + \frac{p(\log 2+\gamma)(\log n)^{p/a-1}}{an^{p/a}}\right) + O\left(\frac{(\log n)^{p/a-2}}{n^{p/a}}\right), & p/a \geq 3, \end{cases}$$

and

$$Var(c_n^*) = \begin{cases} \frac{\zeta(2)-2(\log 2)^2}{b^2 n^2} + O\left(\frac{(\log n)^2}{n^{7/3}}\right), & a=1; \\ O\left(\frac{(\log n)^{2/a-2}}{n^{2/a}}\right), & 1/a \geq 2. \end{cases}$$

PROOF. For a beta(a,b) distribution, $F(x) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\,dt$, where $B(a,b)$ is the beta function. By expanding $(1-t)^{b-1}$ in a Maclaurin series and integrating term-by-term we obtain a series representation $\sum_{k=0}^{\infty} c_k x^{a+k}$ for $F(x)$ whose first term is $\frac{1}{aB(a,b)} x^a$. Inverting this series expression for $F(x)$, where $1/a \in \mathbb{Z}^+$, yields a representation of $Q(x)$ as a series of the form $\sum_{k=1}^{\infty} d_k x^{k/a}$ whose first term is $a^{1/a} B(a,b)^{1/a} x^{1/a}$. Since $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $B(1,b) = 1/b$. The expression for $E[c_n^*]$ then follows from Theorem 5.1, and the expression for the variance follows from that for $E[c_n^*]$. □

Other well-known distributions for which Theorem 5.1 can be applied include the half-normal and the Pareto, and, for certain values of their parameters, the gamma, Weibull, and log-logistic distributions as well.

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