# Deranged Exams

*A student takes an exam consisting of matching questions with $n$ possible answers. Unfortunately, the student is completely unprepared and is forced to guess on every question. Each question has exactly one correct answer, and the student does not use the same answer more than once.*

Among the many questions we can ask about this scenario, here are four:

- What is the number of ways that the student can complete the exam and miss the first $k$ questions? Denote this number by $S_{n,k}$, for $0 \leq k \leq n$.
- What is the number of ways that the student can answer all $n$ questions and have exactly $k$ be correct? Let this be $M_{n,k}$, for $0 \leq k \leq n$.
- What is the number of ways that the student can answer all $n$ questions and have question $k$ be the first correct one? Call this $R_{n,k}$, for $1 \leq k \leq n+1$.
- If there are $n$ possible answers from which to choose, but only $k$ questions on the exam, how many ways can the student complete the exam and get all the questions wrong? Denote this by $H_{n,k}$, for $0 \leq k \leq n$.

The answers to these four questions are closely related. The purpose of this paper is to expose these connections and then to discuss some interesting properties of the $S_{n,k}$.

Note: All of these numbers include the derangement numbers $D_n$ as special cases. (A *derangement* is a permutation of $n$ elements in which none of the elements remains in its original position.) We have $S_{n,n} = M_{n,0} = R_{n,n+1} = H_{n,n} = D_n$. Also, we use the usual convention that there is one permutation of the empty set, so that $S_{0,0} = M_{0,0} = H_{n,0} = 1$.

## Relating the numbers.

We show how $S_{n,k}$, $M_{n,k}$, $R_{n,k}$, and $H_{n,k}$ are all related by defining $S_{n,k}$ in terms of each of the others.

The $M_{n,k}$ numbers were introduced by Montmort in his 1713 essay [**4**]. Here is how they relate to $S_{n,k}$.

**Theorem 1.** *For* $0 \leq k \leq n$,

$$S_{n,k} = \sum_{j=0}^{n-k} \frac{\binom{n-k}{j}}{\binom{n}{k+j}} M_{n,n-k-j}.$$

*Proof.* Let $\sigma$ be an exam counted in $S_{n,k}$. Then the student missed exactly $k + j$ questions on $\sigma$, where $0 \leq j \leq n - k$, namely, the first $k$ questions and then $j$ of the remaining $n - k$. Since $M_{n,n-k-j}$ is the number of exams in which exactly $k + j$ answers are wrong, $M_{n,n-k-j}/\binom{n}{k+j}$ is the number of exams in which the first $k + j$ answers are wrong and all others are correct. Thus the number of exams in which the first $k$ answers and any $j$ of the remaining $n - k$ answers are wrong is $M_{n,n-k-j}\binom{n-k}{j}/\binom{n}{k+j}$. Summing over $j$ yields $S_{n,k}$. □

The $R_{n,k}$ numbers are discussed in Riordan [**6**]. The relationship between $S_{n,k}$ numbers and Riordan's $R_{n,k}$ is as follows.

**Theorem 2.** *For* $0 \le k \le n$, $S_{n,k} = R_{n+1,k+1}$.

*Proof.* The removal of question $k+1$ in an exam counted in $R_{n+1,k+1}$ results in an exam containing $n$ questions in which the first $k$ questions are incorrect. Thus there is a one-to-one correspondence between those exams counted in $R_{n+1,k+1}$ and those counted in $S_{n,k}$. □

This indicates a particularly tight relationship between $S_{n,k}$ and $R_{n,k}$. However, it is not quite the case that, other than indexing, the $S_{n,k}$ numbers are identical to the $R_{n,k}$ numbers: There is no $S_{n,k}$ number that corresponds to $R_{n,n+1}$.

Hanson, Seyffarth, and Weston introduced the $H_{n,k}$ numbers in their paper **[3]**, denoting them $D(n,k,0)$. Our next theorem connects $S_{n,k}$ and these numbers.

**Theorem 3.** *For* $0 \le k \le n$, $S_{n,k} = (n-k)!\, H_{n,k}$.

*Proof.* $H_{n,k}$ is the number of ways to answer only the first $k$ questions on an $n$-question exam and miss them all. Given that the first $k$ questions have been missed, there are $(n-k)!$ ways to assign the $n-k$ remaining answers to the $n-k$ remaining questions. Thus $(n-k)!\, H_{n,k}$ is the number of ways to complete an $n$-question exam and miss the first $k$ questions. By definition, this is also $S_{n,k}$. □

## A formula for the $S_{n,k}$ numbers and an $S_{n,k}$ triangle.

Theorems 1, 2, and 3 establish the connection between our four exam scenario questions. Moreover, they also imply that properties of the $S_{n,k}$ numbers immediately give properties of the $M_{n,k}$, $R_{n,k}$, and $H_{n,k}$ numbers as well. We now derive some properties of the $S_{n,k}$ numbers.

First, we derive an explicit formula for the numbers $S_{n,k}$ via the principle of inclusion and exclusion. Let $A_j$ be the set of all possible $n$-question sets of answers in which question $j$ is answered correctly. Then $S_{n,k} = n! - |A_1 \cup A_2 \cup \cdots \cup A_k|$. The principle of inclusion and exclusion (see, for example, [**7**, p. 107-113]) says that $|A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} (-1)^{i+1} \sum_{1 \le j_1 < \cdots < j_i \le k} |A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}|$. For a particular set of questions $j_1, j_2, \ldots, j_i$, $|A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}|$ is the number of exams in which $j_1, j_2, \ldots, j_i$ are all answered correctly. Since there are no restrictions on the answers to the other $n-i$ questions, we have $|A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}| = (n-i)!$. The number of ways of choosing $i$ questions $j_1, j_2, \ldots, j_i$ from $k$ possible questions is $\binom{k}{i}$. Thus $\sum_{1 \le j_1 < \cdots < j_i \le k} |A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}| = \binom{k}{i}(n-i)!$. Thus $S_{n,k} = n! - \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i}(n-i)! = n! + \sum_{i=1}^{k} (-1)^{i} \binom{k}{i}(n-i)!$, yielding the following theorem.

**Theorem 4.** *For* $0 \le k \le n$,

$$S_{n,k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n-i)!.$$

The numbers $S_{n,k}$ also satisfy a simple recurrence relation. Any exam $\sigma$ counted by $S_{n,k-1}$ falls into one of the following two cases: Either the student answers question $k$ correctly or not. If the student answers question $k$ correctly, then $\sigma$ corresponds to an exam with $n-1$ questions and $n-1$ answers in which the student answers the first $k-1$ questions incorrectly. In this case, $\sigma$ is counted by $S_{n-1,k-1}$. If the student answers question $k$ incorrectly, then $\sigma$ is an exam with $n$ questions and $n$ answers in which the student answers the first $k$ questions incorrectly and is counted by $S_{n,k}$. Therefore, $S_{n,k-1} = S_{n,k} + S_{n-1,k-1}$, and we have the following result.

**Theorem 5.** *For* $1 \le k \le n$, $S_{n,k} = S_{n,k-1} - S_{n-1,k-1}$.

Thus the $S_{n,k}$ numbers satisfy a Pascal-like recurrence relation (with subtraction instead of addition). As with Pascal's triangle, we can construct the $S_{n,k}$ numbers, row by row, in the following manner, starting with row 0, element 0:

1. The first number in row $n$ is $n!$.
2. Any other number in row $n$ is the number to its left minus the number diagonally above it to the left.

Table 1 contains the first several rows of the $S_{n,k}$ triangle.

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 0 | | | | | |
| 2 | 2 | 1 | 1 | | | | |
| 3 | 6 | 4 | 3 | 2 | | | |
| 4 | 24 | 18 | 14 | 11 | 9 | | |
| 5 | 120 | 96 | 78 | 64 | 53 | 44 | |
| 6 | 720 | 600 | 504 | 426 | 362 | 309 | 265 |

Table 1: Triangle of $S_{n,k}$ numbers

The $S_{n,k}$ numbers are not new. Dumont and Randrianarivony [**2**] gave three combinatorial interpretations (including ours), Dickson [**1**] showed that they count the number of nonzero terms in a certain class of determinants, Mundfrom [**5**] used them to address an old question of Cayley about the card game Mousetrap, and they appear in two places in the On-Line Encyclopedia of Integer Sequences [**8**]: as A047920 (by rows read left to right) and A068106 (by rows read right to left).

## More properties of the $S_{n,k}$ numbers.

The first is a generalization of Euler's recurrence relation $D_n = (n-1)(D_{n-1} + D_{n-2})$ for derangements. We give a combinatorial proof.

**Theorem 6.** *For* $1 \leq k \leq n$*,*

$$S_{n,k} = (n-1)S_{n-1,k-1} + (k-1)S_{n-2,k-2}.$$

*(Note:* $S_{n-2,-1}$ *is defined to be 0.) In particular,*

$$S_{n,n} = (n-1)S_{n-1,n-1} + (n-1)S_{n-2,n-2},$$

*which is the derangement recurrence.*

*Proof.* Suppose a student's answer to question 1 on an exam counted by $S_{n,k}$ is the correct answer to question $j$. If $k \geq 1$, $j \neq 1$. We can thus divide the exams counted by $S_{n,k}$ for $k \geq 1$ into three groups: (1) those in which $j$ is one of $\{2, 3, \ldots, k\}$ and for which the student's answer to question $j$ is the correct answer to question 1 (so that the student swapped the correct answers to questions 1 and $j$); (2) those in which $j$ is one of $\{2, 3, \ldots, k\}$ and for which the student's answer to question $j$ is not the correct answer to question 1; and (3) those in which $j$ is one of $\{k + 1, k + 2, \ldots, n\}$.

Case 1: If $j$ is one of $\{2, 3, \ldots, k\}$ and the student swapped the correct answers to questions 1 and $j$, then the remaining questions and answers on the student's exam correspond to an exam containing $n - 2$ questions in which the first $k - 2$ are answered incorrectly. For a fixed $j$ there are $S_{n-2,k-2}$ such exams. Since there are $k - 1$ choices for $j$, there are $(k - 1)S_{n-2,k-2}$ exams in this group.

Case 2: If $j$ is one of $\{2, 3, \ldots, k\}$ and the student's answer to question $j$ is not the correct answer to question 1, then questions $\{2, 3, \ldots, n\}$ and answers $\{1, 2, \ldots, n\} - \{j\}$ correspond (by identifying question $j$ with answer 1) to an exam with $n - 1$ questions in which the first $k - 1$ questions are incorrect. For a fixed $j$ there are $S_{n-1,k-1}$ such exams. With $k - 1$ choices for $j$, there are $(k - 1)S_{n-1,k-1}$ exams in this group.

Case 3: If question $j$ is one of questions $\{k + 1, k + 2, \ldots, n\}$, then, as in the previous case, questions $\{2, 3, \ldots, n\}$ and answers $\{1, 2, \ldots, n\} - \{j\}$ correspond to an exam with $n - 1$ questions in which the first $k - 1$ questions are incorrect. For a fixed $j$ there are $S_{n-1,k-1}$ such exams, and there are $n - k$ ways to choose $j$. Thus there are $(n - k)S_{n-1,k-1}$ exams in this group.

Putting the three cases together, we have

$$S_{n,k} = (k-1)S_{n-2,k-2} + (k-1)S_{n-1,k-1} + (n-k)S_{n-1,k-1}$$
$$= (n-1)S_{n-1,k-1} + (k-1)S_{n-2,k-2}.$$

□

(The $H_{n,k}$ version of Theorem 6 is a special case of Theorem 2 in Hanson, Seyffarth, and Weston [**3**]. It also appears in Dickson's 1878 paper [**1**] in a discussion about determinants. Our combinatorial proof is new.)

The argument behind the relationship between the $S_{n,k}$ numbers and those of Montmort (Theorem 1) yields yet another identity satisfied by the $S_{n,k}$ numbers.

**Theorem 7.** *For* $0 \le k \le n$,

$$S_{n+k,k} = \sum_{j=0}^{n} \binom{n}{j} S_{k+j,k+j} = \sum_{j=0}^{n} \binom{n}{j} D_{k+j}.$$

*Proof.* As stated in the proof of Theorem 1, $M_{n,n-k-j}/\binom{n}{k+j}$ is the number of possible exams in which the first $k+j$ answers are incorrect and all the rest are correct. However, an exam counted in $M_{n,n-k-j}/\binom{n}{k+j}$ corresponds to an exam containing only $k+j$ questions in which none are answered correctly. Thus $M_{n,n-k-j}/\binom{n}{k+j} = D_{k+j} = S_{k+j,k+j}$. (This observation was actually made by Riordan [**6**, p. 59].) Applying Theorem 1 and reindexing yields the theorem. □

Theorem 7 implies that column $k$ of the $S_{n,k}$ numbers is the binomial transform (see, for example, [**9**]) of the derangement numbers shifted $k$ times. (The $H_{n,k}$ version of Theorem 7 appears as Equation 17 in Hanson, Seyffarth, and Weston [**3**].)

Another property of the $S_{n,k}$ concerns an interpretation of the sum of row $n$ in the triangle.

**Theorem 8.** *The sum of the numbers in row $n$ of the $S_{n,k}$ triangle is $(n+1)! - D_{n+1}$.*

*Proof.* By Theorem 2, $S_{n,k}$ is the number of ways a student can complete an exam with $n+1$ questions and have question $k+1$ be the first one that the student answers correctly. Thus $\sum_{k=0}^{n} S_{n,k}$ is the number of ways a student can complete an exam with $n+1$ questions and answer at least one question right, which is also $(n+1)! - D_{n+1}$. □

(Riordan [**6**] uses a generating function argument to prove the $R_{n,k}$ version of Theorem 8.)

With as much structure as they have, the $S_{n,k}$ numbers satisfy many other identities as well. Here are a few:

- $S_{n,k} = (n-k)S_{n-1,k} + kS_{n-1,k-1}$, for $0 \le k \le n-1$. (This is proved for $H_{n,k}$ in [**3**] and is an exercise for $R_{n,k}$ in [**6**].)
- $S_{n,k} = nS_{n-1,k} + kS_{n-2,k-1}$, for $0 \le k \le n-1$. (This is given in the entry for Sequence A047920 in [**8**].)
- $S_{n,k} = nS_{n-1,k-1} - (n-k)S_{n-2,k-1}$, for $1 \le k \le n-1$.
- $S_{n,k} = (n+1)S_{n-1,k} - (n-k-1)S_{n-2,k}$, for $0 \le k \le n-2$.
- $(n-2)S_{n,0} = n(n-1)S_{n-1,1}$, for $n \ge 2$.

Interested readers are invited to find combinatorial proofs for these identities or to discover new ones for themselves.

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