1. Use equilibrium point and nullcline analysis to make a phase portrait for the system

\[
\begin{align*}
\frac{dx}{dt} &= 4x - 2x^2 - xy \\
\frac{dy}{dt} &= xy - 3y
\end{align*}
\]

2. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= x^2 + 3x - 2xy \\
\frac{dy}{dt} &= y^2 - 3y - 2xy
\end{align*}
\]

(a) Find the four equilibrium points for this system.

(b) Show that this is a Hamiltonian system and find a Hamiltonian.

(c) Use linearization and the Hamiltonian to show that one of the equilibrium points is a center and the other three are saddles.

(d) Describe each of the different categories of solution curve for this system.

3. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= -x^3 - x^2 - 2xy \\
\frac{dy}{dt} &= 2x^3y - x - 1
\end{align*}
\]

(a) Find the equilibrium points for this system.

(b) Show that \( x = 0 \) is a solution curve.

(c) Show that \( L(x, y) = xe^x e^{-y^2} \) is a Lyapunov function for each of the two regions \( x < 0 \) and \( x > 0 \).

(d) Use this Lyapunov function to develop at least one interesting claim about the phase portrait for this system.
4. Here is a simple model for the spread of a disease in an isolated population. Let \( T \) represent the total number of individuals, \( S \) represent the number of susceptible individuals, \( I \) represent the number of infected individuals, and \( R \) represent the number of recovered individuals. Thus \( T = S + I + R \). Assume a recovered individual has immunity and cannot be reinfected. Also, assume the disease is not fatal and spreads on a time scale that is short compared with the life span of individuals so we can assume \( T \) is constant. The basic assumptions in the model are

- infection occurs due to interaction between susceptible and infected individuals
- infected individuals recover at a constant percentage rate

We can incorporate these assumptions into a simple model as

\[
\begin{align*}
\frac{dS}{dt} &= -\alpha SI \\
\frac{dI}{dt} &= \alpha SI - \beta I \\
\frac{dR}{dt} &= \beta I
\end{align*}
\]

where \( \alpha \) and \( \beta \) are positive constants. Note that we really need only consider the first two equations since these determine \( S \) and \( I \) and we can then compute \( R = T - S - I \). Also, we need only consider the first quadrant of the \( SI \)-plane.

(a) Explain the connection between the terms \(-\alpha SI\) in the first equation and \(\alpha SI\) in the second equation. Also, explain the connection between the terms \(-\beta I\) in the second equation and \(\beta I\) in the third equation.

(b) Find all equilibrium points in the \( SI \)-plane.

(c) Show that linearization at any of these equilibrium points is not useful.

(d) Show that the quantity \( \alpha I + \alpha S - \beta \ln S \) is constant along solution curves of the system.

(e) Make a phase portrait for the first quadrant of the \( SI \)-plane.

(f) Based on your phase portrait, give a qualitative description of how the disease spreads in a situation that starts with one infected individual amid a large number of susceptible individuals so that the initial point is a small perturbation away from \((T,0)\).

(g) For the situation in (f), consider the specific parameter values \( T = 3000, \alpha = 0.00004, \) and \( \beta = 0.05 \). Add quantitative details to your description of how the disease spreads.

(h) Continuing from (g), suppose we can cut the transmission rate in half to \( \alpha = 0.00002 \). Does this produce a significant change in the spread of the disease (either qualitatively or quantitatively)?