1. Consider the one-parameter family of systems
\[ \frac{d\vec{Y}}{dt} = \begin{bmatrix} a & a \\ 1 & -a \end{bmatrix} \vec{Y} \]
where \( a \) can be any real number. Determine all possible phase portrait types for this family. For each possible phase portrait type, give the corresponding value(s) of \( a \).

(14 points)

2. Consider the differential equation
\[ \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 5y = 0. \]
(a) Find the specific solution satisfying the initial conditions \( y(0) = 4 \) and \( y'(0) = 0 \). (15 points)
(b) Sketch a plot of your solution from (a) and describe the long-term behavior of the solution. (5 points)

3. Find the general solution of the differential equation
\[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 4t + 5e^{-4t}. \]

(15 points)

4. Consider the differential equation
\[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 3e^{-2t}. \]
(a) Explain why there are no particular solutions of the form \( y_p = Ae^{-2t} \). (4 points)
(b) Find a particular solution. (8 points)

There is more on the flip side.
5. Consider the damped harmonic oscillator model

\[ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0 \]

with \( m > 0, b \geq 0, \) and \( k > 0. \) For each of the following parameter value sets, determine if the motion is underdamped, critically damped, or overdamped. Also sketch a representative phase portrait in the \( yv\)-plane. Your phase portraits should be qualitatively correct but need not be quantitatively correct. (8 points each)

(a) \( m = 2, b = 4, k = 5 \)

(b) \( m = 2, b = 5, k = 3 \)

6. Analyze the long-term behavior of the damped harmonic oscillator with external forcing modeled by

\[ \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 10y = 2 \sin(3t) \]

(14 points)

7. Show that

\[ y(t) = c_1 \frac{1}{t^2} + c_2 t^3 + t^3 \ln t \]

is the general solution of the differential equation

\[ t^2 \frac{d^2 y}{dt^2} - 6y = 5t^3 \]

for the interval \((0, \infty)\).

Note: You do not need to show where this formula for \( y(t) \) comes from. (9 points)