**Definition:**
A solution of the initial value problem

\[
\frac{dy}{dt} = f(t, y) \quad \text{with } y = y_0 \text{ for } t = t_0
\]

is a differentiable function defined on an interval \((\alpha, \beta)\) containing \(t_0\) that satisfies both the differential equation and the initial condition.

**Note:**
To distinguish between the unknown \(y\) in the problem statement and a specific candidate for this unknown, we might write \(y = \phi(t)\) and then require

\[
\phi'(t) = f(t, \phi(t)) \quad \text{for all } t \text{ in } (\alpha, \beta) \quad \text{with} \quad \phi(t_0) = y_0.
\]
What is a solution?

Example:

\[ \frac{dy}{dt} = y \quad \text{with } y = 5 \text{ for } t = 0 \]

- One solution is \( y = 5e^t \) for \( t \) in \((-1, 1)\).
- Another solution is \( y = 5e^t \) for \( t \) in \((-\infty, \infty)\).
- We say that this second solution is an extension of the first solution.
- In fact, it is the maximal extension since the domain cannot be extended further.
Theorem (informal):
If $f$ and $\partial f / \partial y$ are continuous for all points in the $ty$-plane near $(t_0, y_0)$, then there is a unique solution to the initial value problem

$$\frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(t_0) = y_0.$$ 

Notes:

• Need to be clear on what “points near $(t_0, y_0)$” means.
• Need to be clear on domain of the solution.
• For existence, need only continuity of $f$. For uniqueness, need continuity of both $f$ and $\partial f / \partial y$. 

Existence and Uniqueness Theorem
An existence-uniqueness theorem: precise

**Theorem:**
If $f$ and $\partial f / \partial y$ are continuous in a rectangle 
\[ \{(t, y) \mid a < t < b, c < y < d\} \] containing $(t_0, y_0)$, then there is a value $\epsilon > 0$ defining an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for which there is a unique solution to the initial value problem

\[ \frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(t_0) = y_0. \]

**Note:**
- Uniqueness means that if $y_1$ and $y_2$ are functions defined for $(t_0 - \epsilon, t_0 + \epsilon)$ and each satisfies the IVP, then

  \[ y_1(t) = y_2(t) \quad \text{for all } t \text{ in } (t_0 - \epsilon, t_0 + \epsilon). \]

- No guarantee about solution or uniqueness extending beyond the interval $(t_0 - \epsilon, t_0 + \epsilon)$. 

Existence and Uniqueness Theorem