Solutions: Integrating a vector field over a surface

4. Compute $\int \int_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and $S$ is the open right circular cylinder of radius 2 and height 6 centered at the origin with axis along the $z$-axis oriented so that area vectors point outward (i.e., away from the $z$-axis).

Solution:

In cylindrical coordinates, the cylinder is described by $r = 2$ for $0 \leq \theta \leq 2\pi$ and $-3 \leq z \leq 3$. Expressing cartesian coordinates in terms of cylindrical coordinates (with $r = 2$), we have

$$x = 2 \cos \theta \quad y = 2 \sin \theta \quad z = z.$$ 

Let $d\vec{r}_1$ be an infinitesimal displacement with $z$ held constant so $dz = 0$ and thus

$$d\vec{r}_1 = (-2 \sin \theta \hat{i} + 2 \cos \theta \hat{j} + 0 \hat{k}) d\theta.$$ 

Let $d\vec{r}_2$ be an infinitesimal displacement with $\theta$ held constant so $d\theta = 0$ and thus

$$d\vec{r}_2 = (0 \hat{i} + 0 \hat{j} + \hat{k}) dz.$$ 

Now compute

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = (2 \cos \theta \hat{i} + 2 \sin \theta \hat{j} + 0 \hat{k}) d\theta dz.$$ 

Note that we can match this result with our geometric intuition that all area element vectors $d\vec{A}$ on this cylinder are horizontal. Along the surface, we have

$$\vec{F} = 2 \cos \theta \hat{i} + 2 \sin \theta \hat{j} + z \hat{k}$$

so

$$\vec{F} \cdot d\vec{A} = 4 \, d\theta dz.$$ 

Putting together these pieces, we get

$$\int \int_S \vec{F} \cdot d\vec{A} = \int_{-3}^{3} \int_{0}^{2\pi} 4 \, d\theta dz = 4 \int_{-3}^{3} \, dz \int_{0}^{2\pi} d\theta = 4(6)(2\pi) = 48\pi.$$ 

5. Compute $\int \int_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and $S$ is the paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 1$ oriented so that area vectors point outward (i.e., away from the $z$-axis).
Solution:

In cylindrical coordinates, the equation of the paraboloid is \( z = r^2 \). The piece of the paraboloid with \( 0 \leq z \leq 1 \) projects onto the disk of radius 1 centered at the origin in the \( z = 0 \) plane so we have \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq 1 \). Expressing cartesian coordinates in terms of cylindrical coordinates, we have

\[
x = r \cos \theta \quad y = r \sin \theta \quad z = r^2.
\]

Let \( d\vec{r}_1 \) be an infinitesimal displacement with \( r \) held constant so \( dr = 0 \) and thus

\[
d\vec{r}_1 = (-r \sin \theta \hat{i} + r \cos \theta \hat{j} + 0 \hat{k}) \, d\theta.
\]

Let \( d\vec{r}_2 \) be an infinitesimal displacement with \( \theta \) held constant so \( d\theta = 0 \) and thus

\[
d\vec{r}_2 = (\cos \theta \hat{i} + \sin \theta \hat{j} + 2r \hat{k}) \, dr.
\]

Now compute

\[
d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = (2r^2 \cos \theta \hat{i} + 2r^2 \sin \theta \hat{j} - r \hat{k}) \, drd\theta.
\]

Note that we can match this result with our geometric intuition that area element vectors \( d\vec{A} \) on the paraboloid that are pointing away from the \( z \)-axis will have negative \( \hat{k} \)-components. Along the surface, we have

\[
\vec{F} = r \cos \theta \hat{i} + r \sin \theta \hat{j} + r^2 \hat{k}
\]

so

\[
\vec{F} \cdot d\vec{A} = (2r^3 \cos^2 \theta + 2r^3 \sin^2 \theta - r^3) \, drd\theta = r^3 \, drd\theta.
\]

Putting together these pieces, we get

\[
\int_S \int \vec{F} \cdot d\vec{A} = \left[ \int_0^{2\pi} \int_0^1 r^3 \, drd\theta \right] = \int_0^{2\pi} \left[ \int_0^1 r^3 \, dr \right] \, d\theta = \left[ \int_0^1 r^3 \, dr \right] \left( \frac{1}{4} \right) = \frac{\pi}{2}.
\]

We could also approach this problem using cartesian coordinates. From \( z = x^2 + y^2 \), we get \( dz = 2x \, dx + 2y \, dy \). Let \( d\vec{r}_1 \) be an infinitesimal displacement with \( x \) held constant so \( dx = 0 \) and thus

\[
d\vec{r}_1 = (0 \hat{i} + \hat{j} + 2y \hat{k}) \, dy.
\]

Let \( d\vec{r}_2 \) be an infinitesimal displacement with \( y \) held constant so \( dy = 0 \) and thus

\[
d\vec{r}_2 = (\hat{i} + 0 \hat{j} + 2x \hat{k}) \, dx.
\]

Now compute

\[
d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = (2x \hat{i} + 2y \hat{j} - \hat{k}) \, dxdy.
\]
Along the surface, we have

\[ \vec{F} = x \hat{i} + y \hat{j} + (x^2 + y^2) \hat{k} \]

so

\[ \vec{F} \cdot d\vec{A} = (2x^2 + 2y^2 - x^2 - y^2) \, dx \, dy = (x^2 + y^2) \, dx \, dy. \]

In cartesian coordinates, the unit disk in the \( z = 0 \) plane onto which the surface projects is described by \(-1 \leq x \leq 1\) and \(-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\). Putting together these pieces, we get

\[
\int_S \vec{F} \cdot d\vec{A} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx.
\]

Perhaps the easiest way to evaluate the iterated integral is to transform to polar coordinates, giving us

\[
\int_S \vec{F} \cdot d\vec{A} = \int_{0}^{2\pi} \int_{0}^{1} r^2 (x^2 + y^2) \, r \, dr \, d\theta.
\]

Note that this is exactly the iterated integral we got using the cylindrical coordinates approach. So, the details of evaluating it are identical and we get

\[
\int_S \vec{F} \cdot d\vec{A} = \frac{\pi}{2}.
\]