Fundamental Theorems
Math 280

Spring 2011
If $F'(x) = f(x)$, then $\int_a^b f(x) \, dx = F(b) - F(a)$.

By substituting, we can also write the conclusion as

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

Note: In the above and following theorems, a hypothesis on continuity of the integrand is omitted in order to focus on other details.
Let $C$ be a curve that starts at $A$ and ends at $B$. If $\vec{\nabla} V = \vec{F}$, then

$$\int_C \vec{F} \cdot d\vec{r} = V(B) - V(A).$$

By substituting, we can also write the conclusion as

$$\int_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A).$$
The Divergence Theorem

Let $D$ be a solid region with the closed surface $S$ as the edge of $D$ and area element vectors $d\vec{A}$ for $S$ oriented outward. If $\vec{\nabla} \cdot \vec{F} = f$, then

$$\iiint_{D} f \, dV = \iint_{S} \vec{F} \cdot d\vec{A}.$$ 

By substituting, we can also write the conclusion as

$$\iiint_{D} (\vec{\nabla} \cdot \vec{F}) \, dV = \iint_{S} \vec{F} \cdot d\vec{A}.$$
Stokes’ Theorem

Let $S$ be a surface with the closed curve $C$ as the edge of $S$. Orient the area element vectors $d\vec{A}$ and the curve $C$ to have a right-hand relation. If $\vec{\nabla} \times \vec{F} = \vec{G}$, then

$$\int\int_S \vec{G} \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{r}.$$ 

By substituting, we can also write the conclusion as

$$\int\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{r}.$$ 

Fundamental Theorems
Green’s Theorem (as a special case of Stokes’ Theorem)

Start with \( \vec{F} = P(x, y) \hat{i} + Q(x, y) \hat{j} + 0 \hat{k} \).

\[
\nabla \times \vec{F} = (0 - 0) \hat{i} - (0 - 0) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}.
\]

**D**: region in the \( xy \)-plane with closed curve \( C \) as edge.
Orient curve \( C \) counterclockwise.
Express area element vectors as \( d\vec{A} = dx \, dy \, \hat{k} \).

\[
(\nabla \times \vec{F}) \cdot d\vec{A} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \cdot dx \, dy \, \hat{k} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy.
\]

Stokes’ Theorem for this case:

\[
\int \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy = \oint_C (P \hat{i} + Q \hat{j}) \cdot d\vec{r}.
\]
Using an alternate notation for line integrals, Green's Theorem can also be written as

\[ \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{C} P \, dx + Q \, dy. \]
All together now

**FTC**

\[
\int_a^b F'(x) \, dx = F(b) - F(a)
\]

**FTC for line integrals**

\[
\int_C \vec{\nabla} V \cdot d\vec{r} = V(B) - V(A)
\]

**Divergence**

\[
\iiint_D (\vec{\nabla} \cdot \vec{F}) \, dV = \iint_S \vec{F} \cdot d\vec{A}
\]

**Stokes’**

\[
\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{r}
\]

**Green’s**

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_C P \, dx + Q \, dy
\]

**Fundamental Theorems**