Idea of differentiability

For \( f : \mathbb{R} \rightarrow \mathbb{R} \)

- Idea: \( f \) is differentiable at \( x_0 \) if zooming in on the graph at \((x_0, f(x_0))\) gives a line
- Definition: \( f \) is differentiable at \( x_0 \) if \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \) exists

For \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)

- Idea: \( f \) is differentiable at \((x_0, y_0)\) if zooming in on the graph at \((x_0, y_0, f(x_0))\) gives a plane
- Definition: ???

It is not enough to know that \( f_x(x_0, y_0) \) and \( f_y(x_0, y_0) \) exist.
Example: a nondifferentiable function

\[ f(x, y) = \begin{cases} 
1 & \text{if } x = 0 \text{ or } y = 0 \\
0 & \text{otherwise} 
\end{cases} \]

Note that \( f_x(0, 0) = 0 \) and \( f_y(0, 0) = 0 \) so the partial derivatives exist for \((0, 0)\) but zooming in on \((0, 0, 1)\) does not give a plane.
A sufficient condition for differentiability

Defining **differentiable** for \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a bit messy.

Can give a sufficient condition for differentiability:

**Theorem:**
If \((x_0, y_0)\) is a point in the domain of a function \( f \) with

(A) \( f \) defined for all points in an open disk centered at \((x_0, y_0)\),
    and

(B) \( f_x \) and \( f_y \) each continuous for all points in that open disk

then \( f \) is differentiable for \((x_0, y_0)\).
Differentiability as a hypothesis for other results

Differentiability is often a condition needed as a hypothesis.

Theorem:
If $f$ is differentiable in an open region containing the point $P$, then

$$\left( \frac{df}{ds} \right)_{P, \hat{u}} = \vec{\nabla} f(P) \cdot \hat{u}. $$