CHAPTER 3

ELEMENTARY FUNCTIONS

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable $z$ that reduce to the elementary functions in calculus when $z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

28. THE EXPONENTIAL FUNCTION

As anticipated earlier (Sec. 13), we define here the exponential function $e^z$ by writing

$$e^z = e^x e^{iy} \quad (z = x + iy),$$

where Euler's formula (see Sec. 6)

$$e^{iy} = \cos y + i \sin y$$

is used and $y$ is to be taken in radians. We see from this definition that $e^z$ reduces to the usual exponential function in calculus when $y = 0$; and, following the convention used in calculus, we often write $\exp z$ for $e^z$.

Note that since the positive $n$th root $\sqrt[n]{e}$ of $e$ is assigned to $e^x$ when $x = 1/n$ ($n = 2, 3, \ldots$), expression (1) tells us that the complex exponential function $e^z$ is also $\sqrt[n]{e}$ when $z = 1/n$ ($n = 2, 3, \ldots$). This is an exception to the convention (Sec. 8) that would ordinarily require us to interpret $e^{1/n}$ as the set of $n$th roots of $e$. 
According to definition (1), \(e^x e^{iy} = e^{x+iy}\); and, as already pointed out in Sec. 13, the definition is suggested by the additive property
\[e^{x_1} e^{x_2} = e^{x_1 + x_2}\]
of \(e^x\) in calculus. That property's extension,
\[(3)\]
\[e^{z_1} e^{z_2} = e^{z_1 + z_2},\]
to complex analysis is easy to prove. To do this, we write
\[z_1 = x_1 + i y_1 \quad \text{and} \quad z_2 = x_2 + i y_2.\]
Then
\[e^{z_1} e^{z_2} = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}) = (e^{x_1} e^{y_1})(e^{x_2} e^{y_2}).\]
But \(x_1\) and \(x_2\) are both real, and we know from Sec. 7 that
\[e^{iy_1} e^{iy_2} = e^{(y_1 + y_2)}.\]
Hence
\[e^{z_1} e^{z_2} = e^{(x_1 + x_2)} e^{(y_1 + y_2)},\]
and, since
\[(x_1 + x_2) + i(y_1 + y_2) = (x_1 + i y_1) + (x_2 + i y_2) = z_1 + z_2,\]
the right-hand side of this last equation becomes \(e^{z_1 + z_2}\). Property (3) is now established.

Observe how property (3) enables us to write \(e^{z_1} e^{z_2} = e^{z_1},\) or
\[(4)\]
\[\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}.\]
From this and the fact that \(e^0 = 1\), it follows that \(1/e^z = e^{-z}\).

There are a number of other important properties of \(e^z\) that are expected. According to Example 1 in Sec. 21, for instance,
\[(5)\]
\[\frac{d}{dz} e^z = e^z\]
everywhere in the \(z\) plane. Note that the differentiability of \(e^z\) for all \(z\) tells us that \(e^z\) is entire (Sec. 23). It is also true that
\[(6)\]
\[e^z \neq 0 \quad \text{for any complex number} \ z.\]
This is evident upon writing definition (1) in the form
\[e^z = re^{i\phi} \quad \text{where} \quad r = e^x \text{ and } \phi = y,\]
which tells us that
\[ |e^z| = e^x \quad \text{and} \quad \text{arg}(e^z) = y + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots). \]

Statement (6) then follows from the observation that \(|e^z|\) is always positive.

Some properties of \(e^z\) are, however, not expected. For example, since
\[ e^{x+2\pi i} = e^x e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1, \]
we find that \(e^z\) is periodic, with a pure imaginary period \(2\pi i\):
\[ e^{z+2\pi i} = e^z. \]

The following example illustrates another property of \(e^z\) that \(e^x\) does not have. Namely, while \(e^x\) is never negative, there are values of \(e^z\) that are.

**EXAMPLE.** There are values of \(z\), for instance, such that
\[ e^z = -1. \]
To find them, we write equation (9) as \(e^x e^{iy} = 1e^{i\pi}\). Then, in view of the statement in italics at the beginning of Sec. 8 regarding the equality of two nonzero complex numbers in exponential form,
\[ e^x = 1 \quad \text{and} \quad y = \pi + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots). \]
Thus \(x = 0\), and we find that
\[ z = (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots). \]

**EXERCISES**
1. Show that
   \(\text{(a)} \exp(2 \pm 3\pi i) = -e^2; \quad \text{(b)} \exp \left( \frac{2 + \pi i}{4} \right) = \sqrt[4]{e} (1 + i); \)
   \(\text{(c)} \exp(z + \pi i) = -\exp z. \)
2. State why the function \(2z^2 - 3 - ze^z + e^{-z}\) is entire.
3. Use the Cauchy–Riemann equations and the theorem in Sec. 20 to show that the function \(f(z) = \exp z\) is not analytic anywhere.
4. Show in two ways that the function \(\exp(z^2)\) is entire. What is its derivative?
   \(\text{Ans.} \quad 2z \exp(z^2). \)
5. Write \(|\exp(2z + i)|\) and \(|\exp(iz^2)|\) in terms of \(x\) and \(y\). Then show that
   \[ |\exp(2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}. \]
6. Show that \(|\exp(z^2)| \leq \exp(|z|^2).\)
7. Prove that $|\exp(-2z)| < 1$ if and only if $\text{Re} z > 0$.

8. Find all values of $z$ such that

   (a) $e^z = -2$;  
   (b) $e^z = 1 + \sqrt{3}i$;  
   (c) $\exp(2z - 1) = 1$.

   Ans.  
   (a) $z = \ln 2 + (2n + 1)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$;  
   (b) $z = \ln 2 + \left(\frac{2n + 1}{3}\right)\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$;  
   (c) $z = \frac{1}{2} + n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

9. Show that $\exp(iz) = \exp(i\bar{z})$ if and only if $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. (Compare Exercise 4, Sec. 27.)

10. (a) Show that if $e^z$ is real, then $\text{Im} z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$.
   
   (b) If $e^z$ is pure imaginary, what restriction is placed on $z$?

11. Describe the behavior of $e^z = e^x e^{iy}$ as (a) $x$ tends to $-\infty$; (b) $y$ tends to $\infty$.

12. Write $\text{Re}(e^{1/z})$ in terms of $x$ and $y$. Why is this function harmonic in every domain that does not contain the origin?

13. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain $D$. State why the functions

   
   \begin{align*}
   U(x, y) &= e^{u(x, y)} \cos v(x, y), \\
   V(x, y) &= e^{u(x, y)} \sin v(x, y)
   \end{align*}

   are harmonic in $D$ and why $V(x, y)$ is, in fact, a harmonic conjugate of $U(x, y)$.

14. Establish the identity

   \[ (e^z)^n = e^{nz} \quad (n = 0, \pm 1, \pm 2, \ldots) \]

   in the following way.

   (a) Use mathematical induction to show that it is valid when $n = 0, 1, 2, \ldots$.
   
   (b) Verify it for negative integers $n$ by first recalling from Sec. 7 that

   \[ z^n = (z^{-1})^m \quad (m = -n = 1, 2, \ldots) \]

   when $z \neq 0$ and writing $(e^z)^n = (1/e^z)^m$. Then use the result in part (a), together with the property $1/e^z = e^{-z}$ (Sec. 28) of the exponential function.

29. THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

   \[ e^w = z \]

for $w$, where $z$ is any nonzero complex number. To do this, we note that when $z$ and $w$ are written $z = re^{i\Theta} (-\pi < \Theta \leq \pi)$ and $w = u + iv$, equation (1) becomes

   \[ e^u e^{iv} = re^{i\Theta}. \]