1. Show that $y(t) = -1$ is the only solution of the initial value problem

$$\frac{dy}{dt} = 4t(1 + y) \quad y(0) = -1.$$  

(8 points)

2. For each of the following, solve the given differential equation or initial-value problem. Express each result in terms of real-valued functions. (14 points each)

(a) $\frac{dy}{dt} = 3t^2(y^2 + 1)$

(b) $y'' - 10y' + 29y = 0, \quad y(0) = 2, \quad y'(0) = 4$

(c) $y'' - 5y' + 6y = 3e^{4t} + \sin t$

(d) $\frac{d}{dt} \vec{y} = A\vec{y}$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

3. Consider the system of equations

$$\frac{dx}{dt} = x^2 + y^2 - 1$$
$$\frac{dy}{dt} = 2xy$$

(a) Show that $(0, 1)$ is an equilibrium point for this system.

(b) Find the relevant linearized system for this equilibrium point.

(c) Analyze the linearized system in enough detail to draw a phase portrait. Sketch a phase portrait for the linearized system.

(d) Determine if the linearized system is relevant to analyzing the original system.

4. Discuss similarities between the structure of the general solution of a nonhomogeneous linear $n^{th}$ order differential equation and the general solution of a nonhomogeneous system of $n$ linear first order differential equations. (8 points)

There is more on the flip side.
5. Consider a species with a natural rate of change in population modeled as proportional to the population. If the population is harvested (think of fishing) at a constant rate \( h \), a reasonable simple model is

\[
\frac{dp}{dt} = ap - h \quad p(0) = p_0.
\]

Here, \( a \), \( h \), and \( p_0 \) are positive constants. (14 points)

(a) Sketch a slope field for \( 0 \leq t \) and \( 0 \leq y \) with enough detail to show all interesting features.

(b) On your slope field, sketch a solution curve for each of the following cases:

(i) \( h < ap_0 \)  
(ii) \( h = ap_0 \)  
(iii) \( h > ap_0 \).

(c) Explain what this model predicts about the population for each of the three cases

(i) \( h < ap_0 \)  
(ii) \( h = ap_0 \)  
(iii) \( h > ap_0 \).

(d) For the case \( h > ap_0 \), find the time \( T \) at which the population becomes 0.