Some theory on linear homogeneous ODEs

We’ll use the following notation:

- $C^n(a,b)$ is the vector space of functions with continuous $n^{\text{th}}$ derivative on the domain $(a,b)$.
- $L = D^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$ where each of the coefficients $a_i$ is a function of the independent variable and $D$ is the differentiation operator.
- $W_S(t)$ is the Wronskian of the set $S = \{f_1(t), f_2(t), \ldots, f_n(t)\}$ defined as

$$W_S(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f'_1(t) & f'_2(t) & \cdots & f'_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix}$$

**Theorem 1.** Let $S = \{f_1(t), f_2(t), \ldots, f_n(t)\}$ be a set of functions in $C^n(a,b)$. If there is a $t_0$ in $(a,b)$ such that $W_S(t_0)$ is nonzero, then $S$ is linearly independent.

**Proof.** Start with the defining equation of linear independence

$$c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t) = \theta(t)$$

where $\theta(t)$ is the zero function. We must show that the only solution is the trivial solution. Differentiate both sides of this equation $n-1$ times to generate a system of equations

$$c_1f'_1(t) + c_2f'_2(t) + \cdots + c_nf'_n(t) = \theta(t)$$
$$c_1f''_1(t) + c_2f''_2(t) + \cdots + c_nf''_n(t) = \theta(t)$$
$$\vdots$$
$$c_1f^{(n-1)}_1(t) + c_2f^{(n-1)}_2(t) + \cdots + c_nf^{(n-1)}_n(t) = \theta(t)$$

The Wronskian $W_S(t)$ is defined as the determinant of the coefficient matrix for this system. Hence, if the Wronskian is nonzero for $t_0$ in $(a,b)$, the system has a unique solution for that value $t_0$. This unique solution must be the trivial solution because the system of equations is homogeneous. Thus, the trivial solution is the only solution for all values of $t$. \qed

We now look at the set of solutions for an $n^{\text{th}}$ order, linear homogeneous differential equation $L[y(t)] = \theta(t)$. We can view the solution set as the null space $\mathcal{N}(L)$, defined as

$$\mathcal{N}(L) = \{y \in C^n(a,b) | L[y] = 0\}.$$

**Theorem 2.** If $a_{n-1}(t), \ldots, a_1(t), a_0(t)$ are continuous for all $t$ in $(a,b)$ and $L$ is defined as above, then the solution set $\mathcal{N}(L)$ is a subspace of $C^n(a,b)$ of dimension $n$. 
Proof. Since \( L \) is a linear transformation, we know that \( \mathcal{N}(L) \) is a subspace of \( C^n(a, b) \) by a standard theorem of linear algebra (for example, see Theorem NSLTS of FCLA). To show that it has dimension \( n \), we will find a basis with \( n \) elements.

To begin, we claim the existence of \( n \) solutions to the O.D.E. by the existence-uniqueness theorem. In particular, pick some \( t_0 \) in \( I \) and let \( h_1(t), h_2(t), \ldots, h_n(t) \) be the solutions that satisfy the following sets of initial conditions

\[
\begin{align*}
  h_1(t_0) &= 1, & h'_1(t_0) &= 0, & \ldots, & h^{(n-1)}_1(t_0) &= 0 \\
  h_2(t_0) &= 0, & h'_2(t_0) &= 1, & \ldots, & h^{(n-1)}_2(t_0) &= 0 \\
  \vdots & & \vdots & & \vdots & \vdots \\
  h_n(t_0) &= 0, & h'_n(t_0) &= 0, & \ldots, & h^{(n-1)}_n(t_0) &= 1
\end{align*}
\]

To prove that \( \{h_1(t), h_2(t), \ldots, h_n(t)\} \) is a basis for \( N(L) \), we must show two things: one, that the set is linearly independent; and two, that the set spans \( N(L) \).

To show linear independence, we note that

\[
W[h_1, h_2, \ldots, h_n](t_0) = 1 \neq 0.
\]

By Theorem 1, the set \( \{h_1(t), h_2(t), \ldots, h_n(t)\} \) is linearly independent.

To prove that the set \( \{h_1(t), h_2(t), \ldots, h_n(t)\} \) spans \( N(L) \), we must show that any other solution in \( N(L) \) can be written as a linear combination of the elements in \( \{h_1(t), h_2(t), \ldots, h_n(t)\} \). Let \( y(t) \) be any solution. At \( t_0 \), this solution and its derivatives have some values

\[
y(t_0) = c_1, \quad y'(t_0) = c_2, \quad \ldots, \quad y^{(n-1)}(t_0) = c_n.
\]

Consider the solution given by the linear combination \( c_1 h_1(t) + c_2 h_2(t) + \cdots + c_n h_n(t) \). Note that at \( t_0 \), this solution and its derivatives have the same values as the solution \( y(t) \) and its derivatives. Hence, by the existence-uniqueness theorem, we have

\[
y(t) = c_1 h_1(t) + c_2 h_2(t) + \cdots + c_n h_n(t).
\]

This gives \( y(t) \) as a linear combination of the elements in \( \{h_1(t), h_2(t), \ldots, h_n(t)\} \) and thus completes the proof.

**Theorem 3.** Let \( S = \{y_1(t), y_2(t), \ldots, y_n(t)\} \) be a set of \( n \) solutions to the \( n \)th order linear differential equation \( L[y] = 0 \) with coefficient functions \( a_i \) that are continuous for \( (a, b) \). The set \( S \) is linearly independent if and only if there is a \( t_0 \) in \( (a, b) \) such that \( W_S(t_0) \) is nonzero.

**Proof.** The proof of one direction follows immediately from Theorem 1. The proof of the other direction is an exercise.

**Exercises**

1. Determine if \( S = \{t^3, |t|^3\} \) is linearly independent in \( C^2(-\infty, \infty) \) without using the Wronskian. Now compute the Wronskian of \( S \). Comment on these results in relation to Theorems 1 and 3.

2. Finish the proof of Theorem 3. Hint: Work with the contrapositive of the statement to be proven: If \( W_S(t) = 0 \) for all \( t \) in \( (a, b) \), then \( S \) is linearly dependent. Don’t forget that here the set \( S \) consists of solutions to \( L[y] = 0 \).