Notes on line integrals

Version 2 (April 22, 2004)

Definition of line integral

We are given a vector field $\vec{F}$ and a curve $C$ in the domain of $\vec{F}$. The general idea of line integral is

$$\text{line integral of } \vec{F} \text{ over curve } C = \text{the limit of a sum of terms each having the form (component of } \vec{F} \text{ tangent to } C)(\text{length of piece of } C).$$

Here’s how we make the idea precise. Break the curve $C$ into $n$ pieces with endpoints $P_1$, $P_2$, ..., $P_{n+1}$. (See Figure 1 at the end.) We can refer to these as $P_i$ with the index $i$ ranging from 1 to $n+1$. Define $\Delta \vec{R}_i$ to be the displacement between point $P_i$ and point $P_{i+1}$. (See Figure 2.) That is, $\Delta \vec{R}_i = \overrightarrow{P_iP_{i+1}}$. At each of the points, compute the vector field output $\vec{F}(P_i)$. Recall that the dot product $\vec{F}(P_i) \cdot \Delta \vec{R}_i$ can be written as

$$\vec{F}(P_i) \cdot \Delta \vec{R}_i = \|\vec{F}(P_i)\| \|\Delta \vec{R}_i\| \cos \theta = \left(\|\vec{F}(P_i)\| \cos \theta\right) \|\Delta \vec{R}_i\|.$$

The last expression shows that this dot product gives the component of $\vec{F}$ tangent to $C$ times the length of a piece of $C$. This is what we want to add up. We define the line integral of $\vec{F}$ for the curve $C$ as the limit of such a sum:

$$\int_C \vec{F} \cdot d\vec{R} = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(P_i) \cdot \Delta \vec{R}_i$$

You can think of $d\vec{R}$ as an “infinitesimal” version of $\Delta \vec{R}_i$. The direction of $d\vec{R}$ is tangent to the curve at each point. (See Figure 3.)

Notation

The text often uses an alternate notation for the line integral. Here’s the connection: Write the vector field $\vec{F}$ in terms of components as $\vec{F} = u \hat{i} + v \hat{j} + w \hat{k}$ and write the vector $d\vec{R}$ in terms of components as $d\vec{R} = dx \hat{i} + dy \hat{j} + dz \hat{k}$. Here, think of $dx$ as a small displacement parallel to the $x$-axis, $dy$ as a small displacement parallel to the $y$-axis, and $dz$ as a small displacement parallel to the $z$-axis. With these component expressions, we can write out the dot product as

$$\vec{F} \cdot d\vec{R} = (u \hat{i} + v \hat{j} + w \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k}) = ud\,dx + vd\,dy + wd\,dz.$$

Using this, the notation for line integral can be written

$$\int_C \vec{F} \cdot d\vec{R} = \int_C u\,dx + v\,dy + w\,dz.$$
The text favors the expression on the right side and I generally use the expression on the left side.

Most of the problems are given using the notation on right side. For example, Problem 7 of Section 13.2 gives the line integral

\[ \int_C (-y \, dx + x \, dy). \]

From this, you can read off that the vector field is \( \vec{F} = -y \, \hat{i} + x \, \hat{j} \).

**Computing line integrals**

In computing line integrals, the general plan is to express everything in terms of a single variable. This is a reasonable thing to do because a curve is a one-dimensional object. The essential things are to determine the form of \( d\vec{R} \) for the curve \( C \) and the outputs \( \vec{F}(P) \) along the curve \( C \), all in terms of one variable. The displacement \( d\vec{R} \) is defined to have components

\[ d\vec{R} = dx \, \hat{i} + dy \, \hat{j} \]

How to proceed depends on how we describe the curve. In general, we have two choices: a relation between the coordinates or a parametric description. The two solutions to the following example show how to work with each of these.

**Example:** Compute the line integral of \( \vec{F}(x, y) = 3 \, \hat{i} + 2 \, \hat{j} \) for the curve \( C \) that is the upper half of the circle of radius 1 traversed from left to right.

**Note:** To get started, you should draw a picture showing the curve and a few of the vector field outputs along the curve.

**Solution 1:** The equation of the circle is \( x^2 + y^2 = 1 \). From this, we compute

\[ 2x \, dx + 2y \, dy = 0. \]

Solving for \( dy \) and substituting from \( x^2 + y^2 = 1 \) gives

\[ dy = -\frac{x}{y} \, dx = -\frac{x}{\sqrt{1-x^2}} \, dx. \]

This is the relation between \( dx \) and \( dy \) for a displacement \( d\vec{R} \) along the circle. Substituting this gives

\[ d\vec{R} = dx \, \hat{i} + dy \, \hat{j} = dx \, \hat{i} - \frac{x}{\sqrt{1-x^2}} \, dx \, \hat{j} = \left( \hat{i} - \frac{x}{\sqrt{1-x^2}} \, \hat{j} \right) \, dx \]

The vector field here is constant so all outputs along the curve \( C \) are \( \vec{F}(P) = 3\hat{i} + 2\hat{j} \). We thus have

\[ \vec{F} \cdot d\vec{R} = \left( 3 \hat{i} + 2 \hat{j} \right) \cdot \left( \hat{i} - \frac{x}{\sqrt{1-x^2}} \, \hat{j} \right) \, dx = \left( 3 - \frac{2x}{\sqrt{1-x^2}} \right) \, dx \]
This is the integrand. For the curve $C$, the variable $x$ ranges from $-1$ to $1$, so we have

$$\int_C \vec{F} \cdot d\vec{R} = \int_{-1}^{1} \left( 3 - \frac{2x}{\sqrt{1-x^2}} \right) dx = \text{some work to be done here} = 6.$$  

**Solution 2:** We parametrize the curve by

$$x = -\cos t \quad \text{and} \quad y = \sin t \quad \text{for} \quad 0 \leq t \leq \pi.$$  

You should confirm that this traces out the curve $C$ in the correct direction (from left to right). From these, we compute

$$dx = \sin t \, dt \quad \text{and} \quad dy = \cos t \, dt.$$  

Substituting into $d\vec{R}$ gives

$$d\vec{R} = dx \, \hat{i} + dy \, \hat{j} = \sin t \, dt \, \hat{i} + \cos t \, dt \, \hat{j} = (\sin t \, \hat{i} + \cos t \, \hat{j}) \, dt.$$  

The vector field here is constant so all outputs along the curve $C$ are $\vec{F}(P) = 3\hat{i} + 2\hat{j}$. We thus have

$$\vec{F} \cdot d\vec{R} = (3\hat{i} + 2\hat{j}) \cdot (\sin t \, \hat{i} + \cos t \, \hat{j}) \, dt = (3\sin t + 2\cos t) \, dt.$$  

This is the integrand. For the curve $C$, the variable $t$ ranges from $0$ to $\pi$, so we have

$$\int_C \vec{F} \cdot d\vec{R} = \int_{0}^{\pi} (3\sin t + 2\cos t) \, dt = \text{some work to be done here} = 6.$$  

**Comments:** In comparing the two solutions, you might think that the algebra is more complicated in Solution 1. This is probably so. The advantage of Solution 1 is that we all know the equation of a circle is $x^2 + y^2 = 1$. For Solution 2, to get started, we need to parametrize the curve. This is not too bad for a circle. The choice of which style to use depends on personal preference and the easiest way to describe a given curve.
Figure 1. The curve $C$ broken into pieces with endpoint $P_i$.

Figure 2. The curve $C$ with the vectors $\Delta \vec{R}_i$ (in red) and $\vec{F}(P_i)$ (in blue).

Figure 3. The curve $C$ with an example of $d\vec{R}$ and $\vec{F}$ at a point.