A Sturm-Liouville problem consists of

ODE: \(-\left( p(x)y'(x) \right)' + q(x)y(x) = \lambda y(x)\) \quad \text{for } a < x < b

BC1: \quad \alpha_1 y(a) + \alpha_2 y'(a) = 0
BC2: \quad \beta_1 y(b) + \beta_2 y'(b) = 0

A Sturm-Liouville problem is regular if

- \(p\ p',\ \text{and } q\ \text{are continuous for } a \leq x \leq b\)
- \(p(x) > 0\ \text{for } a \leq x \leq b\)
- \((\alpha_1, \alpha_2) \neq (0, 0)\ \text{and } (\beta_1, \beta_2) \neq (0, 0)\)
For a regular Sturm-Liouville problem:

1. There are countably many eigenvalues $\lambda_n$, $n = 1, 2, 3, \ldots$
2. Each eigenvalue is real.
3. There is a smallest eigenvalue and no largest eigenvalue so the eigenvalues can be ordered $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lim_{n \to \infty} \lambda_n = \infty$.
4. For each eigenvalue, there is a one-dimensional eigenspace that is spanned by a single eigenfunction $y_n$.
5. The eigenfunction $y_n$ has $n - 1$ zeros for $a < x < b$.
6. The set of eigenfunctions $\{y_n|n = 1, 2, 3, \ldots\}$ is a maximal orthogonal set in $L^2[a, b]$ with respect to the inner product $< f, g > = \int_a^b f(x)g(x) \, dx$ and hence a basis.