Divergence

• Define: divergence of \( \vec{F} \) at \( P \) is the flux density at \( P \):

\[
\text{div } \vec{F}(P) = \lim_{\Delta D \to P} \frac{\int \int \vec{F} \cdot d\vec{A}}{\Delta V}
\]

• Compute: For \( \vec{F} = P \hat{i} + Q \hat{j} + R \hat{k} \), have

\[
\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}.
\]

• Interpret:
If \( \vec{F} \) is fluid flow velocity, can think of div \( \vec{F} \) in the following way: for a given point \((x, y, z)\), the number \( \text{div } \vec{F}(x, y, z) \) gives the rate at which fluid is being injected into the flow at that point.
Divergence Theorem

\[ \iiint_D \vec{\nabla} \cdot \vec{F} \, dV = \iiint_D \frac{\Delta S}{dV} \, dV \]

\[ = \iiint_D \text{flux through surface of infinitesimal piece of } D \, dV \]

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\[ = \text{flux through surface of } D \]

\[ = \oiint_S \vec{F} \cdot d\vec{A} \]
Curl

- Define: \( \hat{n} \)-component of \( \text{curl} \) of \( \vec{F} \) at \( P \) is the circulation density at \( P \):

\[
(curl \vec{F}) \cdot \hat{n} = \lim_{\Delta C \to P} \frac{\Delta C}{\Delta A} (\vec{F} \cdot d\vec{r})
\]

- Compute: For \( \vec{F} = P \hat{i} + Q \hat{j} + R \hat{k} \), have

\[
curl \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} = \nabla \times \vec{F}.
\]

- Interpret:

If \( \vec{F} \) is fluid flow velocity, can think of \( \text{curl} \) \( \vec{F} \) in the following way: for a given point \((x, y, z)\), the direction of the vector \( curl \vec{F}(x, y, z) \) is the direction in which to orient an infinitesimal paddlewheel to get the fastest rotation rate and the magnitude of \( curl \vec{F}(x, y, z) \) is proportional to that rotation rate.
Stokes’ Theorem

$$\int\int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int\int_S (\nabla \times \vec{F}) \cdot \hat{n} \, dA$$

$$= \oint \vec{F} \cdot d\vec{r}$$

$$= \int\int_S \frac{\Delta C}{dA} \, dA$$

= circulation around edge of infinitesimal piece of $S$

$$= \int\int_S \frac{\text{circulation around edge of infinitesimal piece of } S}{dA} \, dA$$

= circulation around edge of $S$

$$= \oint_C \vec{F} \cdot d\vec{r}$$