More on equations of planes in space

So far, we have seen several forms for the equation of a plane:

\[ Ax + By + Cz + D = 0 \]  standard form
\[ z = m_x x + m_y y + b \]  slopes-intercept form
\[ z - z_0 = m_x(x - x_0) + m_y(y - y_0) \]  point-slopes form

Using vectors, we can add another form that is coordinate-free.

A plane can be specified by giving a vector \( \vec{n} \) perpendicular to the plane (called a normal vector) and a point \( P_0 \) on the plane. We can develop a condition or test to determine whether or not a variable point \( P \) is on the plane by thinking geometrically and using the dot product. Here’s the reasoning:

1. \( P \) is on the plane if and only if the vector \( \overrightarrow{P_0P} \) is parallel to the plane.
2. The vector \( \overrightarrow{P_0P} \) is parallel to the plane if and only if \( \overrightarrow{P_0P} \) is perpendicular to the normal vector \( \vec{n} \).
3. The vectors \( \overrightarrow{P_0P} \) and \( \vec{n} \) are perpendicular if and only if their dot product is zero:
\[ \vec{n} \cdot \overrightarrow{P_0P} = 0. \]

So, the condition \( \vec{n} \cdot \overrightarrow{P_0P} = 0 \) is a new form for the equation of a line. We’ll refer to this as the point-normal form. We can see how the point-normal form relates to our familiar forms by introducing coordinates and components. Let \( P_0 \) have coordinates \((x_0, y_0, z_0)\), the variable point \( P \) have coordinates \((x, y, z)\), and the normal vector \( \vec{n} \) have components \(\langle n_x, n_y, n_z \rangle\). With these, the vector \( \overrightarrow{P_0P} \) has components \(\langle x-x_0, y-y_0, z-z_0 \rangle\). So, the point-normal form can be written as

\[ \overrightarrow{P_0P} \cdot \vec{n} = 0 = \langle n_x, n_y, n_z \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle. \]

The last expression is the same as \( Ax + By + Cz + D \) if we identify \( n_x \) as \( A \), \( n_y \) as \( B \), \( n_z \) as \( C \) and \(- (n_x x_0 + n_y y_0 + n_z z_0) \) as \( D \). This is perhaps easier to see in an example.

**Example:** Find the standard form for the equation of the plane that contains the point \((6, 5, 2)\) and has normal vector \(\langle 7, -3, 4 \rangle\).

**Solution:** With \((x, y, z)\) as the coordinates of a variable point, we can write

\[ \overrightarrow{P_0P} \cdot \vec{n} = 0 = \langle 7, -3, 4 \rangle \cdot \langle x - 6, y - 5, z - 2 \rangle \]
\[ = 7(x - 6) - 3(y - 5) + 4(z - 2) \]
\[ = 7x - 42 - 3y + 15 + 4z - 8 \]
\[ = 7x - 3y + 4z - 35. \]

So the standard form of the equation for this plane is \(7x - 3y + 4z - 35 = 0\).
Exercises

1. Use the point-normal equation to determine which, if any, of the following points are on the plane that has normal vector $2\hat{i} - \hat{j} + 6\hat{k}$ and contains the point $(3, 4, 2)$.

(a) $(5, -4, 0)$  
(b) $(1, 6, 2)$  
(c) $(2, 8, 3)$

*Answer:* $(5, -4, 0)$ and $(2, 8, 3)$ are on the plane; $(1, 6, 2)$ is not

2. Find the slopes-intercept form of the equation that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = 5\hat{i} - 3\hat{j} + 2\hat{k}$.

*Answer:* $z = -\frac{5}{2}x + \frac{3}{2}y$

3. Find the slopes-intercept form of the equation for the plane that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = \langle -6, 1, 5 \rangle$.

*Answer:* $z = \frac{6}{5}x - \frac{1}{5}y - \frac{57}{5}$

*Note:* The original version of this problem had $\vec{n} = \langle -6, 1, 0 \rangle$. A plane with this normal vector is vertical (since the normal vector is horizontal) and so does not have a slopes-intercept form equation.

4. Find the standard form of the equation for the plane that contains the point $(6, 3, 0)$ and is parallel to a second plane given by the equation $5x + 2y - 9z = 14$.

*Answer:* $5x + 2y - 9z - 36 = 0$ or $5x + 2y - 9z = 36$

5. Find the standard form of the equation for the plane that contains the point $(7, -2, 1)$ and is perpendicular to the vector from the origin to that same point.

*Answer:* $7x - 2y + z - 54 = 0$ or $7x - 2y + z = 54$