More on equations of planes in space

So far, we have seen several forms for the equation of a plane:

- $Ax + By + Cz + D = 0$  \hspace{1cm} \text{standard form}
- $z = m_x x + m_y y + b$  \hspace{1cm} \text{slopes-intercept form}
- $z - z_0 = m_x (x - x_0) + m_y (y - y_0)$  \hspace{1cm} \text{point-slopes form}

Using vectors, we can add another form that is coordinate-free.

A plane can be specified by giving a vector $\vec{n}$ perpendicular to the plane (called a normal vector) and a point $P_0$ on the plane. We can develop a condition or test to determine whether or not a variable point $P$ is on the plane by thinking geometrically and using the dot product. Here’s the reasoning:

- $P$ is on the plane if and only if the vector $\overrightarrow{P_0P}$ is parallel to the plane.
- The vector $\overrightarrow{P_0P}$ is parallel to the plane if and only if $\overrightarrow{P_0P}$ is perpendicular to the normal vector $\vec{n}$.
- The vectors $\overrightarrow{P_0P}$ and $\vec{n}$ are perpendicular if and only if their dot product is zero:

$$\vec{n} \cdot \overrightarrow{P_0P} = 0.$$ 

So, the condition $\vec{n} \cdot \overrightarrow{P_0P} = 0$ is a new form for the equation of a line. We’ll refer to this as the point-normal form. We can see how the point-normal form relates to our familiar forms by introducing coordinates and components. Let $P_0$ have coordinates $(x_0, y_0, z_0)$, the variable point $P$ have coordinates $(x, y, z)$, and the normal vector $\vec{n}$ have components $\langle n_x, n_y, n_z \rangle$. With these, the vector $\overrightarrow{P_0P}$ has components $\langle x - x_0, y - y_0, z - z_0 \rangle$. So, the point-normal form can be written as

$$0 = \vec{n} \cdot \overrightarrow{P_0P} = \langle n_x, n_y, n_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = n_xx + n_yy + n_zz - (n_xx_0 + n_yy_0 + n_zz_0).$$

The last expression is the same as $Ax + By + Cz + D$ if we identify $n_z$ as $A$, $n_y$ as $B$, $n_z$ as $C$ and $-(n_xx_0 + n_yy_0 + n_zz_0)$ as $D$. This is perhaps easier to see in an example.

**Example:** Find the standard form for the equation of the plane that contains the point $(6, 5, 2)$ and has normal vector $\langle 7, -3, 4 \rangle$.

**Solution:** With $(x, y, z)$ as the coordinates of a variable point, we can write

$$0 = \vec{n} \cdot \overrightarrow{P_0P} = \langle 7, -3, 4 \rangle \cdot \langle x - 6, y - 5, z - 2 \rangle = 7(x - 6) - 3(y - 5) + 4(z - 2) = 7x - 3y + 4z - 42 + 15 - 8 = 7x - 3y + 4z - 35.$$

So the standard form of the equation for this plane is $7x - 3y + 4z - 35 = 0$. 
Exercises

1. Use the point-normal equation to determine which, if any, of the following points are on the plane that has normal vector $2\hat{i} - \hat{j} + 6\hat{k}$ and contains the point $(3, 4, 2)$.
   
   (a) $(5, -4, 0)$  
   (b) $(1, 6, 2)$  
   (c) $(2, 8, 3)$

2. Find the slopes-intercept form of the equation that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = 5\hat{i} - 3\hat{j} + 2\hat{k}$.

3. Find the slopes-intercept form of the equation for the plane that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = \langle -6, 1, 5 \rangle$.

4. Find the standard form of the equation for the plane that contains the point $(6, 3, 0)$ and is parallel to a second plane given by the equation $5x + 2y - 9z = 14$.

5. Find the standard form of the equation for the plane that contains the point $(7, -2, 1)$ and is perpendicular to the vector from the origin to that same point.