Conservative vector fields

Let \( \vec{F} = P \hat{i} + Q \hat{j} + R \hat{k} \) be a vector field that is continuous on some domain in \( \mathbb{R}^n \). Let \( D \) be an open, connected, and simply connected region in the domain of \( \vec{F} \). The following conditions are equivalent (that is, any one of the conditions implies any of the other conditions):

1. \( \oint_C \vec{F} \cdot d\vec{r} = 0 \) for every smooth closed curve \( C \) in \( D \)

2. \( \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \) for every pair of smooth curves in \( C_1 \) and \( C_2 \) in \( D \) that have a common start point and a common end point.

3. There is a function \( V \) such that \( \vec{\nabla} V = \vec{F} \) for all point in \( D \).

4. The following equations hold for all points in \( D \):
   \[
   \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.
   \]

A vector field satisfying any one (and hence all) of these properties is said to be conservative for the region \( D \). The second property is referred to as path-independence. A function \( V \) satisfying the third property with respect to \( \vec{F} \) is called a potential function for the vector field \( \vec{F} \). The fourth property is called the component test.