Limit of a sequence

We want to write down a precise definition of what it means to say a sequence has a limit (and thus is convergent). As example, the sequence \( \{1/n\} = \{1, 1/2, 1/3, 1/4, \ldots\} \) is convergent with the limit 0. We begin with an informal idea.

**Rough idea:** The number \( A \) is the limit of the sequence \( \{a_n\} \) if, as \( n \) gets large, the elements \( a_n \) “settle down” so that \( A \) is the only reasonable value at “the end of the list”.

To make this precise, we quantify what we mean by “large” values of the index \( n \) and we quantify what we mean by “settle down”. We will use \( N \) to denote a specific index value that counts as “large”. We will use \( \epsilon \) to denote a measure of how close \( a_n \) is to \( A \).

**Precise idea:** The number \( A \) is the limit of the sequence \( \{a_n\} \) if for any positive measure \( \epsilon > 0 \), there is an index value \( N \) beyond which all elements \( a_n \) are within \( \epsilon \) of \( A \).

We can use inequalities to express this more compactly (and in a way that is easier to manipulate mathematically). Rather than writing “positive measure \( \epsilon \)”, we use \( \epsilon > 0 \). In place of writing “index value \( N \) beyond which”, we use \( n > N \). Finally, rather than writing “elements \( a_n \) are within \( \epsilon \) of \( A \)” we use \( |a_n - A| < \epsilon \).

**Compact version:** The number \( A \) is the limit of the sequence \( \{a_n\} \) if for any \( \epsilon > 0 \), there is an index value \( N \) so that \( n > N \) implies \( |a_n - A| < \epsilon \).

**Example:** To prove that \( A = 0 \) is the limit of \( \{a_n\} = \{1/n\} \), we start by considering a fixed value of \( \epsilon > 0 \). So \( \epsilon \) is a given from which we need to construct (or show the existence of) an appropriate value of \( N \). We need to find \( N \) to guarantee that \( n > N \) implies \( |a_n - A| < \epsilon \). In this case, we need \( |1/n - 0| < \epsilon \). This is equivalent to \( n > 1/\epsilon \). So, any integer bigger than \( 1/\epsilon \) will work as a value of \( N \). To be specific, we can choose \( N \) to be the smallest integer that is larger than \( 1/\epsilon \).

So, given any \( \epsilon > 0 \), we choose \( N \) to be the smallest integer larger than \( 1/\epsilon \) to have \( n > 1/\epsilon \). If \( n > N \), then \( 1/n < 1/N < \epsilon \). So \( 1/n < \epsilon \) which is equivalent to \( |1/n - 0| < \epsilon \). Therefore 0 is the limit of \( \{1/n\} \).