Using antidifferentiation to get a power series representation

In this example, we will get a power series representation for \( \tan^{-1} x \) by antidifferentiating a known power series representation. Since

\[
\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}
\]

we know that

\[
\tan^{-1} x = \int \frac{1}{1 + x^2} \, dx + C. \tag{2}
\]

We also know that

\[
\frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \ldots \quad \text{for } -1 < x < 1. \tag{3}
\]

(Recall that we got this result by substituting \( u = -x^2 \) into the geometric series \( \frac{1}{1-u} = \sum_{k=0}^{\infty} u^k \).)

Substituting this series representation for \( \frac{1}{1 + x^2} \) into (2), we get

\[
\tan^{-1} x = \int \sum_{k=0}^{\infty} (-1)^k x^{2k} \, dx + C. \tag{4}
\]

Theorem 20 in the text allows us to interchange the order of summation and integration in this to give

\[
\tan^{-1} x = \sum_{k=0}^{\infty} \int (-1)^k x^{2k} \, dx + C. \tag{5}
\]

We can move the constant factors out of the integral to get

\[
\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} + C. \tag{6}
\]

From the power rule, we know \( \int x^{2k} \, dx = \frac{1}{2k+1} x^{2k+1} \). Using this in (6) gives us

\[
\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} + C. \tag{7}
\]

To evaluate the constant term \( C \), we note that \( \tan^{-1} 0 = 0 \). Thus \( C = 0 \). So, we have

\[
\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1} = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \ldots \tag{8}
\]

Further analysis reveals that this equality is valid for \(-1 \leq x \leq 1\).