Determine whether the series \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)} \) converges absolutely, converges conditionally, or diverges.

First, look at absolute convergence by analyzing the series

\[
\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{n \ln(n)} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}
\]

We know that the harmonic series \( \sum_{n=2}^{\infty} \frac{1}{n} \) diverges. The additional factor of \( \ln(n) \) in the denominator will only contribute very slow growth so we might conjecture that \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \) also diverges. A comparison test to \( \sum_{n=0}^{\infty} \frac{1}{n^p} \) is not useful so we instead look at the related improper integral:

\[
\int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x \ln(x)} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{\ln(u)} \right]_{2}^{b} \text{ using } dx = \frac{1}{u} \, du
\]

\[
= \lim_{b \to \infty} \left[ \ln(u) \right]_{2}^{b} = \lim_{b \to \infty} \left[ \ln(b \ln(b)) - \ln(2 \ln(2)) \right] = \infty.
\]

So \( \int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx \) diverges and therefore so does \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \).

Thus, \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)} \) does not converge absolutely.

We can use the Alternating Series Test to check if \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)} \) converges. With \( u_n = \frac{1}{n \ln(n)} \), we check the three hypotheses:

1. \( u_n > 0 \) for \( n \geq 2 \) since \( n > 0 \) and \( \ln(n) > 0 \) for \( n \geq 2 \).
2. \( u_{n+1} < u_n \) for \( n \geq 2 \) since \( n < n+1 \) and \( \ln(n) < \ln(n+1) \).
3. \( u_n \to 0 \) since \( n \to \infty \) and \( \ln(n) \to \infty \).

Since all three hypotheses hold for \( u_n = \frac{1}{n \ln(n)} \), the conclusion holds and thus \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)} \) converges.

Since \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)} \) converges but not absolutely, it converges conditionally.