Some theory on systems of first-order linear homogeneous ODEs

We’ll use the following notation:

- \( C^1_n(a, b) \) is the vector space of column vector functions (of size \( n \)) with continuous first derivatives on the interval \((a, b)\)
- \( \vec{\theta}(t) \) is the zero column vector function (i.e., the column vector function for which each entry is identically zero for all \( t \) in \((a, b)\))
- \([\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n]\) is the matrix with column vectors \( \vec{A}_i \)
- \( W_S(t) \) is the Wronskian of the set \( S = \{\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)\} \) defined as
  \[
  W_S(t) = \det[\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)]
  \]

**Theorem 1.** Let \( S = \{\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)\} \) be a set of functions in \( C^1_n(a, b) \). If there is a \( t_0 \) in \((a, b)\) such that \( W_S(t_0) \) is nonzero, then \( S \) is linearly independent.

**Proof.** Start with the defining equation of linear independence

\[
  c_1\vec{f}_1(t) + c_2\vec{f}_2(t) + \cdots + c_n\vec{f}_n(t) = \vec{\theta}(t).
\]

We must show that the only solution is the trivial solution. First we introduce some notation. Let \( F(t) = [\vec{f}_1(t), \vec{f}_2(t), \ldots, \vec{f}_n(t)] \). Let \( \vec{c} = [c_1, c_2, \ldots, c_n]^T \). We can then write the defining equation as

\[
  F(t)\vec{c} = \vec{\theta}(t).
\]

The Wronskian \( W_S(t) \) is defined as the determinant of the coefficient matrix for this system. Since the Wronskian is nonzero for \( t_0 \) in \((a, b)\), the system has a unique solution for that value \( t_0 \). This unique solution must be the trivial solution because the system of equations is homogeneous. Thus, the trivial solution is the only solution for all values of \( t \). \(\square\)

We now look at the set of solutions for a homogeneous system of \( n \) linear first-order differential equations.

**Theorem 2.** If \( A(t) \) is an \( n \times n \) matrix function that is continuous on the interval \((a, b)\), then the solution space \( S = \{\vec{y} \in C^1_n(a, b) \mid \frac{dy}{dt} = A\vec{y}\} \) is a subspace of \( C^1_n(a, b) \) with dimension \( n \).
Proof. It is straightforward to show that that $S$ is a subspace of $C^1_n(a, b)$. One could do this directly or one could show that $\frac{d}{dt} - A(t)$ is a linear operator and recognize that $S$ is the null space of this operator. To show that $S$ has dimension $n$, we will find a basis with $n$ elements.

To begin, we claim the existence of $n$ solutions to the system by the existence-uniqueness theorem. In particular, pick some $t_0$ in $(a, b)$ and let $\vec{h}_1(t), \vec{h}_2(t), \ldots, \vec{h}_n(t)$ be the solutions which satisfy the initial conditions

$$\vec{h}_i(t_0) = \vec{e}_i$$

where $\vec{e}_i$ denotes the $i$th column of the $(n \times n)$ identity matrix $I_n$. Let $B = \{\vec{h}_1(t), \vec{h}_2(t), \ldots, \vec{h}_n(t)\}$. To prove that $B$ is a basis for $S$, we must show two things: one, that $B$ is linearly independent; and two, that $B$ spans $S$.

To show linear independence, we note that $W_B(t_0) = \det(I_n) = 1 \neq 0$. By Theorem 1, the set $B$ is linearly independent.

To prove that the set $B$ spans $S$, we must show that any other solution in $S$ can be written as a linear combination of the elements in $B$. Let $\vec{y}(t)$ be any solution. For $t_0$, this solution has some value

$$\vec{y}(t_0) = \vec{c}$$

where $\vec{c} = [c_1, c_2, \ldots, c_n]^T$. Consider the solution given by the linear combination $c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + \cdots + c_n\vec{h}_n(t)$. Note that at $t_0$, this solution has the same value as the solution $\vec{y}(t)$. Hence, by the existence-uniqueness theorem, we have

$$\vec{y}(t) = c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + \cdots + c_n\vec{h}_n(t)$$

for all $t$ in $(a, b)$. This gives $\vec{y}(t)$ as a linear combination of the elements in $B$ and thus completes the proof. \qed