Some theory on linear homogeneous ODEs

We’ll use the following notation:

- $C^n(a,b)$ is the vector space of functions with continuous $n$th derivative on the domain $(a,b)$.
- $L = D^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$ where each of the coefficients $a_i$ is a function of the independent variable and $D$ is the differentiation operator.
- $W_S(t)$ is the Wronskian of the set $S = \{f_1(t), f_2(t), \ldots, f_n(t)\}$ defined as

$$W_S(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f'_1(t) & f'_2(t) & \cdots & f'_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}_1(t) & f^{(n-1)}_2(t) & \cdots & f^{(n-1)}_n(t) \end{bmatrix}$$

**Theorem 1.** Let $S = \{f_1(t), f_2(t), \ldots, f_n(t)\}$ be a set of functions in $C^n(a,b)$. If there is a $t_0$ in $(a,b)$ such that $W_S(t_0)$ is nonzero, then $S$ is linearly independent.

**Proof.** Start with the defining equation of linear independence

$$c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t) = \theta(t)$$

where $\theta(t)$ is the zero function. We must show that the only solution is the trivial solution. Differentiate both sides of this equation $n - 1$ times to generate a system of equations

$$c_1f'_1(t) + c_2f'_2(t) + \cdots + c_nf'_n(t) = \theta(t)$$
$$c_1f''_1(t) + c_2f''_2(t) + \cdots + c_nf''_n(t) = \theta(t)$$
$$\vdots$$
$$c_1f^{(n-1)}_1(t) + c_2f^{(n-1)}_2(t) + \cdots + c_nf^{(n-1)}_n(t) = \theta(t)$$

The Wronskian $W_S(t)$ is defined as the determinant of the coefficient matrix for this system. Hence, if the Wronskian is nonzero for $t_0$ in $(a,b)$, the system has a unique solution for that value $t_0$. This unique solution must be the trivial solution because the system of equations is homogeneous. Thus, the trivial solution is the only solution for all values of $t$. \qed

We now look at the set of solutions for an $n$th order, linear homogeneous differential equation $L[y(t)] = \theta(t)$. We can view the solution set as the null space $\mathcal{N}(L)$, defined as

$$\mathcal{N}(L) = \{y \in C^n(a,b) | L[y] = 0\}.$$

**Theorem 2.** If $a_{n-1}(t), \ldots, a_1(t), a_0(t)$ are continuous for all $t$ in $(a,b)$ and $L$ is defined as above, then the solution set $\mathcal{N}(L)$ is a subspace of $C^n(a,b)$ of dimension $n$. 
Proof. Since $L$ is a linear transformation, we know that $\mathcal{N}(L)$ is a subspace of $C^n(a, b)$ by a standard theorem of linear algebra (for example, see Theorem NSLTS of FCLA). To show that it has dimension $n$, we will find a basis with $n$ elements.

To begin, we claim the existence of $n$ solutions to the O.D.E. by the existence-uniqueness theorem. In particular, pick some $t_0$ in $I$ and let $h_1(t), h_2(t), \ldots, h_n(t)$ be the solutions that satisfy the following sets of initial conditions

$$
h_1(t_0) = 1, \quad h_1'(t_0) = 0, \quad \ldots, \quad h_1^{(n-1)}(t_0) = 0
$$

$$
h_2(t_0) = 0, \quad h_2'(t_0) = 1, \quad \ldots, \quad h_2^{(n-1)}(t_0) = 0
$$

$$
\vdots
$$

$$
h_n(t_0) = 0, \quad h_n'(t_0) = 0, \quad \ldots, \quad h_n^{(n-1)}(t_0) = 1
$$

To prove that $\{h_1(t), h_2(t), \ldots, h_n(t)\}$ is a basis for $N(L)$, we must show two things: one, that the set is linearly independent; and two, that the set spans $N(L)$.

To show linear independence, we note that

$$W[h_1, h_2, \ldots, h_n](t_0) = 1 \neq 0.$$ 

By Theorem 1, the set $\{h_1(t), h_2(t), \ldots, h_n(t)\}$ is linearly independent.

To prove that the set $\{h_1(t), h_2(t), \ldots, h_n(t)\}$ spans $N(L)$, we must show that any other solution in $N(L)$ can be written as a linear combination of the elements in $\{h_1(t), h_2(t), \ldots, h_n(t)\}$. Let $y(t)$ be any solution. At $t_0$, this solution and its derivatives have some values

$$y(t_0) = c_1, \quad y'(t_0) = c_2, \quad \ldots, \quad y^{(n-1)}(t_0) = c_n.$$ 

Consider the solution given by the linear combination $c_1 h_1(t) + c_2 h_2(t) + \cdots + c_n h_n(t)$. Note that at $t_0$, this solution and its derivatives has the same values as the solution $y(t)$ and its derivatives. Hence, by the existence-uniqueness theorem, we have

$$y(t) = c_1 h_1(t) + c_2 h_2(t) + \cdots + c_n h_n(t).$$

This gives $y(t)$ as a linear combination of the elements in $\{h_1(t), h_2(t), \ldots, h_n(t)\}$ and thus completes the proof.

**Theorem 3.** Let $S = \{y_1(t), y_2(t), \ldots, y_n(t)\}$ be a set of $n$ solutions to the $n^{th}$ order linear differential equation $L[y] = 0$ with coefficient functions $a_i$ that are continuous for $(a, b)$. The set $S$ is linearly independent if and only if there is a $t_0$ in $(a, b)$ such that $W_S(t_0)$ is nonzero.

**Proof.** The proof of one direction follows immediately from Theorem 1. The proof of the other direction is an exercise. 

**Exercises**

1. Determine if $S = \{t^3, |t|^3\}$ is linearly independent in $C^2(−\infty, \infty)$ without using the Wronskian. Now compute the Wronskian of $S$. Comment on these results in relation to Theorems 1 and 3.

2. Finish the proof of Theorem 3. Hint: Work with the contrapositive of the statement to be proven: If $W_S(t) = 0$ for all $t$ in $(a, b)$, then $S$ is linearly dependent. Don’t forget that here the set $S$ consists of solutions to $L[y] = 0$. 