A precise definition of limit

Definition:
Let \( f \) be a function whose domain contains a set of the form \( \{x \mid 0 < |x - a| < r\} \) for some \( r \). The number \( L \) is the limit of \( f \) at \( a \) if there is a positive function \( \delta(\varepsilon) \) with domain \((0, \infty)\) such that \( 0 < |x - a| < \delta(\varepsilon) \) implies that \( |f(x) - L| < \varepsilon \).

Notation:
If the number \( L \) is the limit of the function \( f \) at \( a \), we denote this by \( \lim_{x \to a} f(x) = L \).

Comment:
The condition that \( \delta(\varepsilon) \) has domain \((0, \infty)\) says that \( \delta(\varepsilon) \) is defined for any \( \varepsilon > 0 \). Saying that \( \delta(\varepsilon) \) is a positive function means that \( \delta(\varepsilon) > 0 \) for any \( \varepsilon > 0 \). This is equivalent to saying that the range of \( \delta(\varepsilon) \) is contained in the interval \((0, \infty)\).

Example: Prove that \( \lim_{x \to 1} (4x + 1) = 5 \).
Solution: In this case, \( f(x) = 4x + 1 \), \( a = 1 \), and \( L = 5 \).
Let \( \delta(\varepsilon) = \frac{\varepsilon}{4} \). Note that \( \delta(\varepsilon) \) is defined for \( \varepsilon > 0 \) and that \( \delta(\varepsilon) > 0 \) for each \( \varepsilon > 0 \).
Now assume \( 0 < |x - 1| < \delta(\varepsilon) \). Thus \( |x - 1| < \frac{\varepsilon}{4} \). Multiply both sides by 4 to get \( 4|x - 1| < \varepsilon \) or \( |4x - 4| < \varepsilon \). Since \( -4 = 1 - 5 \), we can write the last inequality as \( |4x + 1 - 5| < \varepsilon \). This is equivalent to \( |f(x) - 5| < \varepsilon \).
We have shown that \( \delta(\varepsilon) = \frac{\varepsilon}{4} \) is a positive function with domain \((0, \infty)\) such that \( 0 < |x - 1| < \delta(\varepsilon) \) implies \( |f(x) - 5| < \varepsilon \) for the function \( f(x) = 4x + 1 \). We have thus proven that \( \lim_{x \to 1} (4x + 1) = 5 \).

Example: Prove that \( \lim_{x \to 0} x^2 = 0 \).
Solution: In this case, \( f(x) = x^2 \), \( a = 0 \), and \( L = 0 \).
Let \( \delta(\varepsilon) = \sqrt{\varepsilon} \). Note that \( \delta(\varepsilon) \) is defined for \( \varepsilon > 0 \) and that \( \delta(\varepsilon) > 0 \) for each \( \varepsilon > 0 \).
Now assume \( 0 < |x - 0| < \delta(\varepsilon) \). Thus \( |x| < \sqrt{\varepsilon} \). Square both sides to get \( |x|^2 < \varepsilon \) or \( |x^2| < \varepsilon \). Since \( x^2 = x^2 - 0 \), we can write the last inequality as \( |x^2 - 0| < \varepsilon \). This is equivalent to \( |f(x) - 0| < \varepsilon \).
We have shown that \( \delta(\varepsilon) = \sqrt{\varepsilon} \) is a positive function with domain \((0, \infty)\) such that \( 0 < |x - 0| < \delta(\varepsilon) \) implies \( |f(x) - 0| < \varepsilon \) for the function \( f(x) = x^2 \). We have thus proven that \( \lim_{x \to 0} x^2 = 0 \).

Problems:
1. Prove \( \lim_{x \to 2} (3x - 1) = 5 \).
2. Prove \( \lim_{x \to 4} (6 - 2x) = -2 \).
3. Prove \( \lim_{x \to 5} \frac{x}{10} = \frac{1}{2} \).
4. Prove \( \lim_{x \to 0} x^3 = 0 \).