Density in the calculus sequence

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Places for density in the calculus sequence

- integral calculus: as examples of constructing definite integrals for non-geometric quantities (in place of more traditional work and pressure examples)
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- multivariate calculus: primary motivation/interpretation for double, triple, line, and surface integrals (of scalar-valued functions)
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- integral calculus: as examples of constructing definite integrals for non-geometric quantities (in place of more traditional work and pressure examples)
- multivariate calculus: primary motivation/interpretation for double, triple, line, and surface integrals (of scalar-valued functions)
- start with addressing the conception of density students bring to the calculus sequence
Student conceptions of density

► when asked “What is density?” my students typically respond “Density is mass divided by volume.”
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- from volume to length and area
- from uniform to non-uniform
- start with a handout to introduce these generalizations in two steps
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Step 1: different quantities and different dimensions

- straightforward questions that involve only multiplication

1. The density of aluminum is about 2 g/cm$^3$. Determine the mass of an aluminum cube with sides of length 2 cm.

2. A particular type of rope has a length density of $\lambda = 0.15$ kg per meter. What is the mass of a 3 meter piece of this rope?

3. A standard type of newsprint has an area density of $48.8$ g/m$^2$. Determine the mass of a roll of this newprint that is 2 meters wide and 100 meters long.

4. Bacteria in a circular petri dish are distributed uniformly with a number density of $5.4 \times 10^3$ per cm$^2$. What is the number of bacteria in a dish of radius 5.5 cm?

5. Charge is distributed uniformly on a circular ring with a charge density of $-4.21 \times 10^{-6}$ Coulombs per cm. What is the total charge on a ring of radius 1.2 cm?
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\frac{dQ}{dl} = \lambda \\
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Calculus II project problem

Consider the problem of computing the total number of bacteria in a circular petri dish. The bacteria colony is more dense at the center than at the edges of the petri dish. Let $r$ denote radial distance from the center of the dish measured in centimeters (cm). Let $\sigma$ be the density of the bacteria colony, measured in number per square centimeter (#/cm$^2$). Note that $\sigma$ varies with radius $r$.

(a) Construct a definite integral to compute the total number of bacteria in a petri dish of radius $R$.

(b) Compute the total number of bacteria if the density is $\sigma_0$ at the center of the dish and decreases linearly to zero at the edge of the dish.

(c) Get a numerical value for the total number with the density as in (b) and the values $\sigma_0 = 5 \times 10^3$ per cm$^2$ and $R = 5.5$ cm.
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Solution outline

\[ dm = \sigma \, dA \]

geometry:

\[ dA = 2 \pi r \, dr \]

substitution:

\[ dm = \sigma \, 2 \pi r \, dr \]

summing:

\[ m = \int_{0}^{R} 2 \pi \sigma r \, dr \]
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A rectangular piece of cloth is soaked in dye and then hung vertically to dry. As the cloth dries, the dye flows down so that more ends up at the bottom than at the top. The dried dye has a mass density that varies linearly from zero at the top edge to a maximum value at the bottom edge. Use $H$ for the height of the cloth, $W$ for the width of the cloth, and $\sigma_0$ for the maximum density.
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\[ dm = \sigma \, dA = \sigma \, W \, dh \]

\[ m = \int_0^H \sigma W \, dh = \int_0^H \left( \frac{\sigma_0}{H} \, h \right) W \, dh = \cdots = \frac{1}{2} \sigma_0 WH \]
Charge is distributed on a hemisphere of radius $R$. Think of this as the northern hemisphere of the earth. The area charge density is proportional to the distance from the plane containing the equator with a value of 0 on the equator and a value of $\sigma_0$ at the north pole. Compute the total charge on the hemisphere in terms of $R$ and $\sigma_0$. 

Calculus III project problem

A hydrogen atom consists of one proton and one electron. A free hydrogen atom is one that experiences no external forces. In a free hydrogen atom, the electron can be in one of infinitely many discrete states. These states are labeled by three integers, usually denoted \( n, l, \) and \( m \). For each state, there is an electron location probability density that gives the probability density (per volume) for the location of the electron as a function of position (measured with respect to the proton).

The \( n = 3, l = 2, m = 0 \) state of a free hydrogen atom has an electron probability density (per volume) given by

\[
\rho(r, \phi, \theta) = \frac{1}{39366\pi} r^4 e^{-2r/3} (3 \cos^2 \phi - 1)^2
\]

where \((r, \phi, \theta)\) are spherical coordinates as we use them in class. The origin of the coordinate system is the location of the proton. The radial coordinate \( r \) is measured in units of Bohr radii where the Bohr radius is equal to about \( 5.3 \times 10^{-11} \) meters. (So, for example, \( r = 2 \) means a radial distance of 2 Bohr radii.)
Calculus III project problem (continued)

1. Compute an expression for the probability of finding the electron between \( r = a \) and \( r = b \) for a hydrogen atom in this state.
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2. Compute the probability of finding the electron in each of the unit intervals \( r = k \) to \( r = k + 1 \) for \( k \) between 0 and 25. Identify the interval in which the electron is most likely to be found.
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3. Compute the probability of finding the electron anywhere in space. Does this result make sense?
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- using density as a theme for integral calculus and multivariable calculus provides context for applications targeting students outside physics & engineering

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- using density provides a framework for motivating all flavors of integral (single, double, triple, line, surface) for scalar-valued functions

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