Solution by Jesse Jenks
Problem 7.30

Problem Statement: Use Lebesque’s criterion to prove the following.

(a) If $f$ is of bounded variation on $[a, b]$, then $f \in R$ on $[a, b]$.

(b) If $f \in R$ on $[a, b]$ and $g$ is any subinterval $[c, d] \subset [a, b]$, $|f| \in R$ and $f^2 \in R$ on $[a, b]$.

(c) If $f \in R$ and $g \in R$ where $g$ is bounded away from 0 on $[a, b]$, then $f/g \in R$.

(d) If $f$ and $g$ have the same discontinuities on $[a, b]$, then $f \in R$ if $g \in R$.

(e) If $g \in R$ is bounded by $m \leq g \leq M$, and $f$ is continuous on $[m, M]$, then $h = f(g) \in R$.

Proof. Lebesque’s criterion states that if $f$ is defined and bounded on $[a, b]$ and $D$ is the set of discontinuities of $f$ in $[a, b]$, then $f \in R$ if and only if $D$ has measure 0.

(a) By theorem 6.2, $f$ has a countable number of discontinuities, and by theorem 7.44, any countable subset of $\mathbb{R}$ has measure 0, so $f$ satisfies Lebesque’s criterion, and thus $f \in R$.

(b) If $f \in R$ on $[a, b]$, then the set of discontinuities $D_{a,b}$ of $f$ has measure 0 and $D_{a,b} \subset [a, b]$, so for any subinterval, the set of discontinuities $D_{c,d} \subset [a, b]$, $D_{c,d} \subset [a, b]$. The number of discontinuities of $|f|$ can have at most a countable number of discontinuities which $f$ does not have. So by theorem 7.44, $|f| \in R$. Now $f^2$ has the same number or fewer discontinuities than $f$, so $f^2 \in R$. Finally, if $g \in R$, the number of discontinuities of $f \cdot g$ is $D_g \cup D_f$, and again by theorem 7.44, $f \cdot g \in R$.

(c) Since $g$ is bounded away from 0, $f/g$ is defined on $[a, b]$. Similar to the case of $f \cdot g$, the number of discontinuities of $f/g$ is $D_f \cup D_g$, so $f/g \in R$.

(d) Suppose $f \in R$ but $g \not\in R$. Then the discontinuities of $g$ would not have measure 0, while the discontinuities of $f$ would have measure 0. Since $f$ and $g$ have the same discontinuities, this is a contradiction, so $g \in R$. The proof of the converse is similar.

(e) Since $f$ is continuous, $f \circ g$ has the same discontinuities as $g$. Since $g \in R$, $h \in R$. 

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