Solution by Jesse Jenks
Problem 7.14

**Problem Statement:** Assume \( f \in R(\alpha) \) on \([a,b]\), where \( \alpha \) is of bounded variation on \([a,b]\). Let \( V(x) \) denote the total variation of \( \alpha \) on \([a,x]\) for \( x \in (a,b) \), and let \( V(a) = 0 \). Show that

\[
|\int_a^b f \, d\alpha| \leq \int_a^b |f| \, dV \leq MV(b)
\]

where \( M \) is an upper bound of \(|f|\) on \([a,b]\).

**Lemma 1.** If \( V(x) \) is the variation of \( \alpha \) on \([a,x]\), then \( \int_a^b |f| dV > |\int_a^b f \, d\alpha| \).

**Proof.** Note that \( V_k = V(x_k) - V(x_{k-1}) \) is the total variation of \( \alpha \) on the interval \([x_{k-1}, x_k]\). Now if we consider the simplest partition of this interval, \( \{x_{k-1}, x_k\} \), we see that

\[
V(x_{k-1}, x_k) \geq |\alpha(x_k) - \alpha(x_{k-1})|.
\]

This implies that for any partition,

\[
\sum_{k=1}^n |f| \Delta V_k \geq \sum_{k=1}^n |f| \Delta \alpha_k
\]

\[
= \sum_{k=1}^n |f| \Delta \alpha_k
\]

\[
\geq \sum_{k=1}^n |f| \Delta \alpha_k \text{ by the triangle inequality.}
\]

Since these inequalities hold for any partition, so

\[
\int_a^b |f| \, dV \geq |\int_a^b f \, d\alpha|
\]

which proves the lemma. \( \square \)

**Proof.** Since \( \alpha \) is of bounded variation, by theorem 6.12, \( V - \alpha \) is an increasing function. So by theorem 7.24,

\[
|\int_a^b f \, d\{V - \alpha\}| \leq \int_a^b |f| \, d\{V - \alpha\}
\]

\[
= \int_a^b |f| \, dV - \int_a^b |f| \, d\alpha.
\]

This implies that \( 0 \leq \int_a^b |f| dV - \int_a^b |f| d\alpha \), and thus

\[
\int_a^b |f| \, d\alpha \leq \int_a^b |f| \, dV.
\]
Now since $|f(x)| \leq M$ for all $x \in [a, b]$, and $V$ is a positive increasing function,

$$\int_a^b |f| \, dV \leq \int_a^b M \, dV$$

$$= M \int_a^b dV$$

$$= M (V(b) - V(a))$$

$$= MV(b).$$

From these inequalities and Lemma 1, we get

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, dV \leq MV(b)$$

which proves the theorem. \qed