Problem Statement: Given a decreasing sequence of real numbers \( \{G(n)\} \) such that \( G(n) \to 0 \) as \( n \to \infty \) Define a function \( f \) on \( S = [0,1] \) in terms of \( \{G(n)\} \) as follows: \( f(0) = 1 \); if \( x \) is irrational, then \( f(x) = 0 \); if \( x \) is the rational \( m/n \) (in lowest terms), then \( f(m/n) = G(n) \). Compute the oscillation \( \omega_f(x) \) at each \( x \) in \( [0,1] \) and show that \( f \in R \) on \( [0,1] \).

Proof. Let \( f \) be defined as,
\[
 f(x) = \begin{cases} 
 1 & \text{if } x = 0; \\
 0 & \text{if } x \text{ irrational; } \\
 G(n) & \text{if } x \text{ rational.} 
\end{cases}
\]

Now, without loss of generality, construct an ball in \( S \), \( B_r x \), centered at some \( x \). The oscillation of \( f \) on \( S \) at that point is,
\[
\Omega_f(B(x;r) \cap S) \to \sup(f(x) - f(y)),
\]
First let \( x \) be rational, then by the definition of \( f \), \( f(x) = G(n) \). The largest rational that could be contained within the ball is the rational at \( x \). All other rationals in \( B_r x \) will have denominators that approach infinity, and by the definition of \( G(n) \) those points will go to 0. The smallest point that could be contained in \( B_r x \) is any irrational, thus,
\[
\Omega_f^{x \text{ rational}} = G(n) - 0 \\
\Omega_f^{x \text{ irrational}} = G(n),
\]
for any rational.

Next let \( x \) be irrational. In this case, again the smallest value this function can have is 0, however the greatest value this function can have is also 0. All rationals that are close enough to the given irrational must have large values in the denominator, thus they all go to 0. That means that
\[
\Omega_f^{x \text{ irrational}} = 0,
\]
for any irrational.

Finally let \( x = 0 \). The same logic applies as before for any rational or irrational constructed at \( x = 0 \). However since the point is centered around 0, by the definition of \( f \),
\[
\Omega_f = 1.
\]
Next to calculate \( \omega_f(x) \) take the limit of \( \Omega_f \) as \( r \) goes to 0,
\[
\omega_f x = \lim_{r \to 0} \Omega_f
\]
Thus \( \omega_f x = 0 \) for \( x \) irrational, \( \omega_f x = 1 \) when \( x = 0 \) and \( \omega_f x = G(n) \) when \( x \) rational.

Finally, to show that \( f \in R \), since \( f = 0 \) at all irrationals, \( f \) is continuous at the irrationals. \( f = G(n) \) at all rationals, however, since the measure of the rationals is zero, the number of discontinuities is countable, thus \( f \in R \).