Solution by Jared Polonitza
Problem 7.12

**Problem Statement:** Give an example of a bounded function $f$ and an increasing function $\alpha$ defined on $[a, b]$ such that $|f| \in R(\alpha)$, but for which $\int_a^b f \, d\alpha$ does not exist.

**Proof.** Let $\alpha = x$ and let $f$ be defined as,

$$f(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
-1 & \text{if } x \in [0, 1] \cap Q^c.
\end{cases}$$

Defining $f$ as such will create a function with an uncountable number of discontinuities, however, $|f|$ converges to 1. Now check Riemann’s condition to see if $\int_a^b f \, d\alpha$ does not exist. Riemann’s condition is given as,

$$0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$ 

By the definition of Riemann’s condition,

$$M_k(f) = \sup \{f(x)\} \quad \text{and} \quad m_k = \inf \{f(x)\}$$

Therefore $M_k = 1$ and $m_k = -1$. Then by Riemann’s condition

$$0 \leq 1 + (-1) < \epsilon$$

$$0 \leq 2 < \epsilon.$$ 

Since this equation is not satisfied for every epsilon $\int_a^b f \, d\alpha$ does not exist. \qed