Problem 7.6 Statement: Use Euler’s summation formula, or integration by parts in a Stieltjes integral, to derive the following identities:

a) \[
\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{n^{s-1}} + s \int_{1}^{n} \frac{x-[x]}{x^{s+1}} \, dx \quad \text{if } s \neq 1.
\]

b) \[
\sum_{k=1}^{n} \frac{1}{k} = \log n - \int_{1}^{n} \frac{x-[x]}{x^2} \, dx + 1.
\]

Proof. Note that

\[
\sum_{k=1}^{n} \frac{1}{k^s} = 1 + \sum_{k=2}^{n} \frac{1}{k^s}.
\]

Now let \( f(x) = \frac{1}{x^s} \), then \( f'(x) = \frac{-s}{x^{s+1}} \). Since \( f' \) is continuous for \( x \neq 0 \) and therefore on the interval \([1, n] \), Euler’s summation formula gives

\[
1 + \sum_{k=2}^{n} \frac{1}{k^s} = 1 + \int_{1}^{n} \frac{1}{x^s} \, dx + \int_{1}^{n} \frac{-s}{x^{s+1}} (x-[x]) \, dx + 1(1-[1]) = \frac{1}{n^s}(n-[n])
\]

\[
= 1 + \int_{1}^{n} \frac{1}{x^s} \, dx + \int_{1}^{n} \frac{s}{x^{s+1}} \, dx + s \int_{1}^{n} \frac{x-[x]}{x^{s+1}} \, dx
\]

Since \( 1, n \in \mathbb{Z} \), so \([1] = 1\) and \([n] = n\)

\[
= 1 + \int_{1}^{n} \frac{1}{x^s} \, dx - s \int_{1}^{n} \frac{1}{x^{s+1}} \, dx + s \int_{1}^{n} \frac{x-[x]}{x^{s+1}} \, dx
\]

\[
= 1 + (1-s) \int_{1}^{n} \frac{1}{x^s} \, dx + s \int_{1}^{n} \frac{x-[x]}{x^{s+1}} \, dx.
\]

Now since \( s \neq 1 \), the antiderivative of \( \frac{1}{x^s} \) can be obtained through the reverse of the power rule. Therefore

\[
1 + (1-s) \int_{1}^{n} \frac{1}{x^s} \, dx + s \int_{1}^{n} \frac{x-[x]}{x^{s+1}} \, dx = 1 + (1-s) \left( \frac{1}{x^{s+1}} \right)_{1}^{n} + s \int_{1}^{n} \frac{x}{x^{s+1}} \, dx
\]

\[
= 1 + \left( \frac{1}{n^{s-1}} - 1 \right) + s \int_{1}^{n} \frac{x}{x^{s+1}} \, dx
\]

\[
= \frac{1}{n^{s-1}} + s \int_{1}^{n} \frac{x}{x^{s+1}} \, dx.
\]

So the first equality is proved. To prove the second equality, note

\[
\sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k}.
\]
Now let \( f(x) = \frac{1}{x} \), then \( f'(x) = -\frac{1}{x^2} \). Since \( f' \) is continuous for \( x \neq 0 \) and therefore on the interval \([1, n]\), Euler’s summation formula gives

\[
1 + \sum_{k=2}^{n} \frac{1}{k} = 1 + \int_1^n \frac{1}{x} dx + \int_1^n \frac{1}{x^2} (x - [x]) dx + 1(1 - [1]) - \frac{1}{n} (n - [n])
\]

\[
= 1 + \int_1^n \frac{1}{x} dx - \int_1^n \frac{x - [x]}{x^2} dx
\]

\[
= 1 + \log n - \log 1 - \int_1^n \frac{x - [x]}{x^2} dx
\]

\[
= 1 + \log n - \int_1^n \frac{x - [x]}{x^2} dx.
\]

Thus the equality is proved. \qed