Problem Statement: A function $f$ satisfies a uniform Lipschitz condition of order $\alpha > 0$ on $[a, b]$ if $|f(x) - f(y)| < M|x - y|^\alpha$ for every $x, y$ in $[a, b]$. Show that a) if $\alpha > 1$, then $f$ is constant, and if $\alpha = 1$, $f$ is of bounded variation, b) give an example of a function that satisfies a uniform Lipschitz condition of order $\alpha < 1$ but is not of bounded variation, and c) give an example of a function that is of bounded variation but does not satisfy any uniform Lipschitz condition.

a) Show that if $\alpha > 1$, then $f$ is constant, and if $\alpha = 1$, $f$ is of bounded variation.

Proof. Let $f$ be a function satisfying a uniform Lipschitz condition of order $\alpha > 1$. Then for $x \in (a, b)$, $|f(x + \epsilon) - f(x)| < M|x + \epsilon - x|^\alpha$ for some $\epsilon$. Since $\alpha > 1$,

$$f'(x) = \lim_{\epsilon \to 0} \frac{|f(x + \epsilon) - f(x)|}{|\epsilon|} \leq \lim_{\epsilon \to 0} Me^{\alpha - 1} = 0,$$

which implies that $f$ is constant. If $\alpha = 1$, then

$$f'(x) = \lim_{\epsilon \to 0} \frac{|f(x + \epsilon) - f(x)|}{|\epsilon|} \leq \lim_{\epsilon \to 0} M,$$

so by theorem 6.6, $f$ is of bounded variation.

b) Let $x_{k+1} = x_k + 2^{1-\alpha}$ with $x_0 = 0$ and $0 < \alpha < 1$. Then $\{x_k\}$ is Cauchy and converges. Now let $x_k = 2^{-k}$, and let

$$f(x) = \begin{cases} |x - x_k|^\alpha, & \text{if } x_k < x \leq (x_k + x_{k+1})/2 \\ |x_{k+1} - x|^\alpha, & \text{if } (x_k + x_{k+1})/2 < x \leq x_{k+1}, \end{cases}$$

Then $f$ satisfies a uniform Lipschitz condition of order $\alpha$ since $|b^\alpha - a^\alpha| < |b - a|^\alpha$ for $0 < a < b < 1$. Now consider the partition $\{t_{2k-1} = x_k, t_{2k} = (x_k + x_{k+1})/2\}$. That is, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 3 + 2^{1-\alpha}$, and so on. Then

$$f(t_{2k}) = |(x_k + x_{k+1})/2 - x_k|^\alpha = |(x_{k+1} - x_k)/2|^\alpha = |(2^{-1})/2|^\alpha = |2^{-1/\alpha}|^\alpha = |k^{-1/\alpha}|^\alpha = |k^{-1}|$$

while

$$f(t_{2k-1}) = 0.$$

This means that $|\Delta f_k| = \frac{1}{k}$, which implies that

$$V_f(a, b) \geq \sum_{k=1}^{\infty} |f(t_k)| = 2 \sum_{k=1}^{\infty} \frac{1}{k},$$

which does not converge, and so $f$ is not of bounded variation.
c Let \( f \) be defined as

\[
f(x) = \begin{cases} 
  x, & \text{if } 1 \leq x < 2 \\
  x + c & \text{if } 2 \leq x \leq 3,
\end{cases}
\]

where \( c > 1 \). Then \( f \) is of bounded variation on \([1, 3]\). Now let \( x = 2 + \epsilon/2 \) and \( y = 2 - \epsilon/2 \) with \( 0 < \epsilon < 1 \). Then \( |x - y| = \epsilon \), but \( |f(x) - f(y)| > c \), so for \( \alpha > 0 \), \( |f(x) - f(y)| \not\leq |x - y|^\alpha \). This shows that \( f \) does not satisfy any uniform Lipschitz condition on \([1, 3]\).