Solution by Henry Woody

Problem 6.11

**Problem 6.11 Statement:** Prove that every absolutely continuous function on $[a, b]$ is continuous and of bounded variation.

**Proof.** To show that $f$ is continuous on $[a, b]$, let $p \in (a, b)$. Then, since $f$ is absolutely continuous, for $(x_1, p), (p, x_2) \subset [a, b]$, such that $|x_1 - p| = |x_2 - p|$, for $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$(p - x_1) + (x_2 - p) < \delta$$

then

$$|f(p) - f(x_1)| + |f(x_2) + f(p)| < \epsilon.$$

In other words, for any $x, p \in (a, b)$, if $|x - p| < \delta/2$ then $|f(x) - f(p)| < \epsilon$. Therefore $f$ is continuous on $(a, b)$. Similarly, since $f$ is absolutely continuous, for $\epsilon > 0$ there exists $\delta > 0$ such that if $(y - a) < \delta$ then $|f(y) - f(a)| < \epsilon$ for some $y \in (a, b]$. Hence $f$ is continuous at $a$, and is similarly continuous at $b$. Thus $f$ is continuous on $[a, b]$.

To show that $f$ is of bounded variation on $[a, b]$, consider a partition $P = \{a = x_0, x_1, x_2, \ldots, b = x_n\}$ of $[a, b]$. Then, since $f$ is absolutely continuous on $[a, b]$, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \epsilon,$$

if

$$\sum_{k=1}^{n} (x_k - x_{k-1}) < \delta.$$

Therefore $f$ is of bounded variation on $[a, b]$. ☐