Problem Statement: Given a function $f$ defined and having a finite derivative $f'$ in the half open interval $(0,1]$, such that $|f'(x)| < 1$. If we define $a_n = f\left(\frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$, then $\lim_{n \to \infty} a_n$ exists.

Proof. Let $i, j \in \mathbb{N}$. By the Mean Value Theorem, we have

$$|a_i - a_j| = \left| f\left(\frac{1}{i}\right) - f\left(\frac{1}{j}\right) \right|$$

$$= \left| f'(c) \left| \frac{1}{i} - \frac{1}{j} \right| \right|$$

$$\leq \left| \frac{1}{i} - \frac{1}{j} \right|$$

$$\leq \frac{1}{i}.$$

Now suppose $(N \leq i) \in \mathbb{N}$, and let $N = \frac{2}{\epsilon}$ where $\epsilon > 0$. Consider

$$|a_i - a_j| \leq \frac{1}{N}$$

$$\leq \frac{\epsilon}{2}$$

$$< \epsilon,$$

from which it follows that $\{a_n\}$ is a Cauchy sequence. Thus, $\lim_{n \to \infty} a_n$ exists. $\square$