Chapter 5: Derivatives

Definition 1
Let \( f \) be defined on \((a, b)\) and assume that \( c \in (a, b) \). Then \( f \) is said to be differentiable at \( c \) whenever the limit
\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]
exists. This limit is called the derivative of \( f \) at \( c \), and is denoted \( f'(c) \).

Classroom Activity 1: Prove that the derivative of \( f(x) = x^2 + 1 \) exists at \( c = 2 \).

Classroom Activity 2: Write down three other common ways of denoting \( f'(c) \).

Theorem 1. If \( f \) is defined on \((a, b)\) and differentiable at \( c \in (a, b) \), then there exists a function \( f^*_c(x) \) which is continuous at \( c \) and which satisfies the equation
\[
f(x) - f(c) = (x - c)f^*_c(x),
\]
for all \( x \in (a, b) \), with \( f^*(c) = f'(c) \). Conversely, if there is a function \( f^* \), continuous at \( c \), which satisfies this equation, then \( f \) is differentiable at \( c \) and \( f'(c) = f^*_c(c) \).

Classroom Activity 3: Sketch \( f^*_c(x) \) if \( f(x) = x + 3 \) and \( c = 1 \).
Classroom Activity 4: Sketch $f'(x)$ if $f(x) = x^2 + 3$ and $c = 1$.

Classroom Activity 5: Prove this theorem.

Theorem 2. If $f$ is differentiable at $c$, then $f$ is continuous at $c$.

Classroom Activity 6: Prove this theorem.

Theorem 3. Assume $f$ and $g$ are defined on $(a, b)$ and differentiable at $c$. Then $f \cdot g$ is also differentiable at $c$, and $(f \cdot g)'(c) = f(c)g'(c) + f'(c)g(c)$.

Classroom Activity 7: Prove this theorem.
**Theorem 4.** Let \( f \) be defined on an open interval \( S \), let \( g \) be defined on \( f(S) \), and consider the composition 
\[
(g \circ f)(x) = g[f(x)].
\]
Assume there is a point \( c \) in \( S \) such that \( f(c) \) is an interior point of \( f(S) \). If \( f \) is differentiable at \( c \) and if \( g \) is differentiable at \( f(c) \), then \( g \circ f \) is differentiable at \( c \) and we have
\[
(g \circ f)'(c) = g'[f(c)]f'(c).
\]

**Classroom Activity 8:** If \( f \) is continuous on all of \( S \), is \( f(c) \) necessarily an interior point of \( f(S) \) under the given hypotheses? Why or why not?

**Classroom Activity 9:** Using your book if necessary, provide an outline to the proof of this theorem.