Chapter 3

COMPUTATIONAL METHODS FOR OPTIMIZATION

The preceding chapters have discussed some of the analytic techniques for solving optimization problems. These techniques form the basis for most optimization models. In this chapter we will study some of the computational problems that arise in real applications and discuss a few of the most popular methods for dealing with them.

3.1 One Variable Optimization

Even for simple one-variable optimization problems, the task of locating global extreme points can be exceedingly difficult. Real problems are usually messy. Even when the functions involved are differentiable everywhere, the computation of the derivative is often complicated. The worst part, however, is solving the equation $f'(x) = 0$. The plain and simple fact is that most equations cannot be solved analytically. The best we can do in most instances is to find an approximate solution by graphical or numerical techniques.

Example 3.1. Reconsider the pig problem of Example 1.1, but now take into account the fact that the growth rate of the pig is not constant. Assume that the pig is young, so that the growth rate is increasing. When should we sell the pig for maximum profit?

We will use the five-step method. Step 1 will be to modify the work we did in Section 1.1, as summarized in Fig. 1.1. Now we cannot simply assume that $u = 200 + 5t$. What would be a reasonable assumption to represent an increasing rate of growth? There are, of course, many possible answers to this question. Let us suppose for now that the growth rate of the pig is proportional
to its weight. In other words, let us assume that

$$\frac{dw}{dt} = cw.$$  (3.1)

From the fact that \(\frac{dw}{dt} = 5 \text{ lbs/day} \) when \(w = 200 \text{ lbs} \), we conclude that \(c = 0.005 \). This leaves us with a simple differential equation to solve for \(w \), namely

$$\frac{dw}{dt} = 0.025w, \quad w(0) = 200.$$  (3.2)

We can solve Eq. (3.2) by separation of variables to obtain

$$w = 200e^{0.025t}.$$  (3.3)

Since all of our other assumptions are unchanged from what was presented in Fig. 1.1, this concludes step 1.

Step 2 is to select our modeling approach. We will model the problem as a one-variable optimization problem. The general solution procedure for one-variable optimization problems was outlined in Section 1.1. In this section we will explore some computational methods that can be used to implement this general solution procedure. Computational methods such as those we present here are often needed in real problems when calculations become too hard or too tedious to perform by hand.

Step 3 is to formulate the model. The only difference between the present case and the problem formulation of Section 1.1 is that we have to replace the weight equation \(w = 200 + 5t \) by Eq. (3.3). This leads to the new objective function

$$g = f(x) = (0.05 - 0.01x)(200e^{0.025x}) - 0.45x.$$  (3.4)

and our problem is to maximize the function in Eq. (3.4) over the set \(S = \{x : x \geq 0\}\).

Step 4 is to solve the model. We will use the graphical method. Good graphing utilities for personal computers, and graphing calculators, are widely available. We start our graphical analysis of this problem by producing a graph of the function in Eq. (3.4) on the same scale as Fig. 1.2, our graph of the original objective function. In this case we are left with the feeling that there is more to see on the graph of this function. We would say that Fig. 3.1 is not a complete graph of this function over the set \(S = [0, \infty)\).

Figure 3.2 is a complete graph. It shows all of the important features we need for the solution of our problem.

How do we know when we have a complete graph? There is no simple answer to this question. Graphing is an exploratory technique. You need to experiment and use good judgment. Of course, we do not need to look at negative values of \(x \), but we also need not look beyond \(x = 65 \). After this point our formula says that the price for pigs is negative, which is clearly nonsensical.

![Graph of net profit \(f(x)\) from (3.4) versus time to sell \(x\) for the pig problem with nonlinear weight model.](image1)

![Complete graph of net profit \(f(x)\) from (3.4) versus time to sell \(x\) for the pig problem with nonlinear weight model.](image2)
3.1. ONE VARIABLE OPTIMIZATION

Let us examine the sensitivity of the optimum coordinates in Eq. (3.5) to the growth rate $c = 0.025$ for the young pig. One way to do this would be to repeat our graphical analysis for several different values of the parameter $c$. However, this would be tedious. We would prefer a more efficient method.

Let us begin by generalizing the model. Now we are assuming that

$$\frac{dx}{dt} = cw, ~ x(0) = 200,$$

so that

$$w = 200e^c.$$  

(3.6)

(3.7)

This leads to the objective function

$$f(x) = (0.65 - 0.01x)(200e^c) - 0.46.$$

(3.8)

From our graphical analysis, we know that for $c = 0.025$, the optimum occurs at an interior critical point, at which point $f'(x) = 0$. Since $f$ is a continuous function of $c$, it seems reasonable to conclude that the same holds for values of $c$ near 0.025. In order to locate this interior critical point, we need to compute the derivative $f'(x)$ and solve the equation $f'(x) = 0$. The first part of this process (computing the derivative) is relatively easy. There is a standard method for computing derivatives that you learned in one-variable calculus. It can be applied to virtually any differentiable function. For complicated expressions the derivative can also be computed using a computer algebra system (Maple,
3.1. ONE VARIABLE OPTIMIZATION

Algorithm: NEWTON’S METHOD

Variables: \( x(n) \) = approximate location of root after \( n \) iterations.
\( N \) = number of iterations

Input: \( x(0), N \)

Process: Begin
for \( n = 1 \) to \( N \) do
Begin
\( x(n) \leftarrow x(n - 1) - \frac{F(x(n))}{F'(x(n - 1))} \)
End
End

Output: \( x(N) \)

Figure 3.5: Pseudocode for Newton’s Method in one variable.

<table>
<thead>
<tr>
<th>( c )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.022</td>
<td>11.603549</td>
</tr>
<tr>
<td>0.023</td>
<td>14.510929</td>
</tr>
<tr>
<td>0.024</td>
<td>17.110574</td>
</tr>
<tr>
<td>0.025</td>
<td>19.486159</td>
</tr>
<tr>
<td>0.026</td>
<td>21.000681</td>
</tr>
<tr>
<td>0.027</td>
<td>23.550565</td>
</tr>
<tr>
<td>0.028</td>
<td>25.332247</td>
</tr>
</tbody>
</table>

Table 3.1: Sensitivity of best time to sell \( c \) to growth rate parameter \( c \) for the pig problem with nonlinear weight model.

In our problem we want to use Newton’s method to find a root of the equation
\[
F(x) = 200e^{0.65} - 0.01x - 2e^{0.65} - 0.45 = 0
\]  
(3.11)
For values of \( c \) near 0.025, we expect to find a root near the point \( x = 19.5 \). We used a computer implementation of Newton’s method to produce the results in Table 3.1. For each value of \( c \), we performed \( N = 10 \) iterations starting at the point \( x(0) = 19.5 \). An additional sensitivity run with \( N = 15 \) was used to verify the accuracy of our results.

Notice that our method of solution for Eq. (3.11) has two steps. First, we apply a global method (graphing) to locate an approximate solution. Then, we
apply a fast local method to determine the exact solution to the desired accuracy. These are the two stages of numerical solution, and they are common to most graphically useful solution methods. For one-variable optimization, graphing is the simplest and most useful global method. Newton’s method is easy to program, and built-in numerical equation solvers are also available on most graph-
ing calculators, spreadsheets, and computer algebra systems. While the details vary, most of these solvers are based on some variation of Newton's method. These solvers can be used safely and effectively in the same manner as Newton's method. First, use a global method to approximate the root. Most solvers require either a point estimate or an interval estimate of the root. Then, exercise the solver and verify the result, either by performing sensitivity analysis on the available tolerance parameter(s), or by substituting the numerical solution back into the original equation. One word of caution: Casual and uncontrolled faith in the results of numerical solvers is dangerous. For many real problems, including some of the exercises in this book, inappropriate use of numerical solvers has been known to produce significant errors. Initial application of an appropriate global method, and subsequent verification of the root, are important parts of the numerical solution procedure. Some calculators, computer algebra systems, and spreadsheets also have numerical optimizers. Usually, these routines apply a variant of Newton's method based on a numerical approximation of the derivative. The same advice applies to these routines. Use a global method to approximate the optimum, exercise the numerical optimizer, and then perform sensitivity analysis on the tolerance parameters to ensure accuracy.

In order to relate our sensitivity results back to the original data in the problem, in Figure 3.6 we have plotted the root \( x \), which represents the optimal time to sell, against the growth rate

\[ y = 200c, \]

which was originally given as \( y = 5 \) lbs/day.

To obtain a numerical estimate of sensitivity, we solve Eq. (3.11) once more, setting \( c = 0.02525 \) (a 1% increase). The solution found was \( x = 20.021136 \), which represents a 2.84% increase in \( x \). Thus, we estimate that \( S(x, c) = 2.84 \).

Since \( y = 200c \), we can easily show that

\[ S(x, y) = S(x, c) = 2.84. \]

Also, if we let \( h \) denote the initial weight of the pig (we assumed \( h = 200 \) lbs), then since

\[ h = 5/c, \]

we have

\[ S(x, h) = \frac{dx}{dh} \frac{h}{x} = \left( \frac{dx}{dc} \frac{5}{dh} \frac{c}{x} \right) = -S(x, c) = -2.84. \]

In fact, it is generally true that if \( y \) is proportional to \( z \), then

\[ S(x, y) = S(x, z), \]

and if \( y \) is inversely proportional to \( z \), then

\[ S(x, y) = -S(x, z). \]

Now we have computed the sensitivity of \( x \) to both the initial weight of the pig and the growth rate of the pig. The other sensitivities are unchanged from the original problem considered in Chapter 1, since the other parameters appear in the objective function in the same manner as before.

The fact that our optimal solution in the present case differs slightly from the results obtained in Chapter 1 also merits a closer examination. Some of these issues of robustness are addressed in the exercises at the end of this chapter. What we can say now is this: If the pig's rate of growth does not diminish, and if the price decline does not accelerate, then we should hold onto the pig for another week. At that time, we can reevaluate the situation on the basis of new data.
3.2 Multivariable Optimization

The practical problems associated with locating the global optimum of a function of several variables are similar in many ways to those discussed in the preceding section. Additional complications arise because of the dimension of the problem. Graphical techniques are not available in dimensions $n > 3$, and solving $\nabla f = 0$ becomes more complicated as the number of independent variables increases. Constrained optimization is also more difficult because the geometry of the feasible region can be more complicated.

**Example 3.2.** A suburban community intends to replace its old fire station with a new facility. The old station was located at the historical city center. City planners intend to locate the new facility more scientifically. A statistical analysis of response-time data yielded an estimate of $3.2 + 1.7\sqrt{x}$ (in minutes) required to respond to a call $x$ miles away from the station. (The derivation of this formula is the subject of Exercises 17 and 18 in Chapter 8.) Estimates of the frequency of calls from different areas of the city were obtained from the fire chief. They are presented in Figure 3.7. Each block represents one square mile, and the numbers inside each block represent the number of emergency calls per year for that block. Find the best location for the new facility.

We will represent locations on the city map by coordinates $(x, y)$, where $x$ is the distance in miles to the west side of town, and $y$ is the distance in miles to the south side. For example, $(1, 0)$ represents the lower left-hand corner of the grid, $(0, 0)$ represents the lower left-hand corner of the map, and $(0, 6)$ represents the upper left-hand corner. For simplicity, we will divide the city into nine 2 × 2-mile squares and assume that each emergency call is located at the center of the square. If $(x, y)$ is the location of the new fire station, the average response time to a call is $z = f(x, y)$, where

$$z = 3.2 + 1.7[8\sqrt{(x-1)^2 + (y-5)^2} + 8\sqrt{(x-3)^2 + (y-5)^2} + 8\sqrt{(x-5)^2 + (y-5)^2} + 21\sqrt{(x-1)^2 + (y-3)^2} + 6\sqrt{(x-3)^2 + (y-3)^2} + 3\sqrt{(x-5)^2 + (y-3)^2} + 18\sqrt{(x-1)^2 + (y-1)^2} + 18\sqrt{(x-3)^2 + (y-1)^2} + 6\sqrt{(x-5)^2 + (y-1)^2}] / 84.$$  

Figure 3.8: 3-D graph of the objective function $f$ over the feasible region.

The problem is to minimize $z = f(x, y)$ over the feasible region $0 \leq x \leq 6, 0 \leq y \leq 6$.

Figure 3.8 shows a 3-D graph of the objective function $f$ over the feasible region. It appears as though $f$ attains its minimum at the unique interior point at which $\nabla f = 0$. Figure 3.9 shows a contour plot of the level sets of $f$, indicating that $\nabla f = 0$ near the point $x = 2$ and $y = 3$.

Now, it is certainly possible to compute $\nabla f$ in this problem, but it is not possible to solve $\nabla f = 0$ algebraically. Further graphical analysis is possible, but it is especially cumbersome for functions of more than one variable. What is needed here is a simple global method for estimating the minimum.