Mathematical Modeling
Third Edition

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Mathematical Modeling
Third Edition
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## Preface

Mathematical modeling is the link between mathematics and the rest of the world. You ask a question. You think a bit, and then you refine the question, phrasing it in precise mathematical terms. Once the question becomes a mathematization question, you use mathematics to find an answer. Then finally (and this is the part that too many people forget), you have to reverse the process, translating the mathematical solution back into a comprehensible, no-nonsense answer to the original question. Some people are fluent in English, and some people are fluent in calculus. We have plenty of each. We need more people who are fluent in both languages and are willing and able to translate. These are the people who will be influential in solving the problems of the future.

This text, which is intended to serve as a general introduction to the area of mathematical modeling, is aimed at advanced undergraduate or beginning graduate students in mathematics and closely related fields. Formal prerequisites consist of the usual freshman–sophomore sequence in mathematics, including one-variable calculus, multivariable calculus, linear algebra, and differential equations. Prior exposure to computing and probability and statistics is useful, but is not required.

Unlike some textbooks that focus on one kind of mathematical model, this book covers the broad spectrum of modeling problems, from optimization to dynamical systems to stochastic processes. Unlike some other textbooks that assume knowledge of only a semester of calculus, this book challenges students to use all of the mathematics they know (because that is what it takes to solve real problems).

The overwhelming majority of mathematical models fall into one of three categories: optimization models, dynamic models, and probability models. The type of model used in a real application might be dictated by the problem at hand, but more often, it is a matter of choice. In many instances, more than one type of model will be used. For example, a large Monte Carlo simulation model may be used in conjunction with a smaller, more tractable deterministic dynamic model based on expected values.

This book is organized into three parts, corresponding to the three main categories of mathematical models. We begin with optimization models. A five-step method for mathematical modeling is introduced in Section 1 of Chapter 1, in the context of one-variable optimization problems. The remainder of the first chapter is an introduction to sensitivity analysis and robustness. These
fundamentals of mathematical modeling are used in a consistent way throughout to master them as well. Chapter 2, on univariate optimization, introduces the method of Lagrange multipliers and provides the benefit of these students in the section on sensitivity analysis for problems with constraints, we learn variables. This sets the stage for our discussion of linear programming later that was added in the second edition. Here we give a practical introduction to connection between linear and integer programming problems, which allows an earlier introduction to the important issue of linear versus continuous models. In this section on linear models, we introduce variables, and equilibrium of state space, state space, and what is done here. Nonlinear dynamical systems in both discrete and continuous systems are intimately connected to time are covered. There is less emphasis on exact analytical solutions in place of these models admit no analytic solution, the second edition. We use both analytic and simulation methods to explore the behavior of linear and continuous dynamical systems, to understand how practical and accessible procedures are used to handle such systems under certain conditions. This section provides a with a sensitive dependence to initial conditions, period doubling, and strange attractors that are fractal sets. Most important, these mathematical curiosities emerge from the study of real-world problems. Finally, in the last part of the book, we introduce probability models. No prior knowledge of probability is assumed. Instead we build upon the material intuitive way as it relates to real-world problems.

Each chapter in this book is followed by a set of challenging exercises. These exercises require significant effort, as well as a certain amount of creativity, on real problems. They were not designed to illustrate the use of any particular mathematical techniques in this book because the problem domain is to understand the fundamentals of linear programming. Although typically overemphasized in applying mathematical modeling to solve real problems. For most students, story problems present plenty of challenge. This book teaches students how to solve story problems. There is a general method that can be applied successfully by any reasonably capable student to solve any story problem. It appears in Chapter 1, Section 1. This same general method is applied to problems of all kinds throughout the text.

Following the exercises in each chapter is a list of suggestions for further reading. This list includes references to a number of UMAP modules in applied mathematics that are relevant to the material in the chapter. UMAP modules can provide interesting supplements to the material in the text, or extra credit projects. All of the UMAP modules are available at a nominal cost from the Consortium for Mathematics and Its Applications (www.comap.com).

One of the major themes of this book is the use of appropriate technology for solving mathematical problems. Computer algebra systems, graphics, and numerical methods all have their place in mathematics. Many students have not had an adequate introduction to these tools. In this course we introduce modern technology in context. Students are motivated to learn because the new technology provides a more convenient way to solve real-world problems. Computer algebra systems and 2-D graphics are useful throughout the course. Some 3-D graphics are used in Chapters 2 and 3 in the sections on multivariable optimization. Students who have already been introduced to 3-D graphics should be encouraged to use what they know. Numerical methods covered in the text include, among others, Newton’s method, linear programming, the Euler method, and linear regression.

The text contains numerous computer-generated graphs, along with instructions on the appropriate use of graphing utilities in mathematics. Computer algebra systems are used extensively in those chapters where significant algebraic calculation is required. The text includes computer output from the computer algebra systems Maple and Mathematica in Chapters 2, 4, 5, and 8. The chapters on computational techniques (Chapters 3, 6, and 9) discuss the appropriate use of numerical algorithms to solve problems that admit no analytic solution. Sections 3.3 and 3.4 on linear integer programming include computer output from the popular linear programming package LINDO. Sections 8.3 and 8.4 on linear regression and time series include output from the commonly used statistical package Minitab.

Students are encouraged to use computer software to assist in calculations. Computer software packages that can be used in conjunction with this textbook.
Computer Algebra Systems:
- Derive, Soft Warehouse, Inc., www.derive.com
- Maple, Waterloo Maple, Inc., www.maplesoft.com
- Mathcad, Mathsoft, Inc., www.mathsoft.com

Statistical Packages:
- Minitab, Minitab, Inc., www.minitab.com
- SAS, SAS Institute, Inc., www.sas.com
- SPSS, SPSS Inc., www.spss.com
- S-PLUS, Insightful Corp., www.insightful.com

Linear Programming Packages:
- LINDO, LINDO Systems, Inc., www.lindo.com
- MPL, Maximal Software, Inc., www.maximal-usa.com
- AMPL, AMPL Optimization, LLC, www.ampl.com
- GAMS, GAMS Development Corp., www.gams.com

The numerical algorithms in the text are presented in the form of pseudo-code. Some instructors will prefer to have students implement the algorithms on their own. On the other hand, if students are not going to be required to use software, All of the algorithms in the text have been implemented on a variety of computer platforms that can be made available to users of this textbook at no additional cost. If you are interested in obtaining a copy, please contact the author or go to www.stt.ssu.edu/~sczub/ modeling.html where you can download these implementations. Also, if you are willing to share your own implementation with other instructors and students, please send us a copy. With your permission, we will make copies available to others at no charge.

The third edition of this text reflects the insightful comments and suggestions of a number of students and faculty. Most significant are two new sections that were widely requested. At the end of Chapter 7, Introduction to Probability Models, is a new section on diffusion. Here we give a gentle introduction to partial differential equations by focusing on the diffusion equation. We provide a simple derivation of the point source solution to this partial differential equation, using Fourier analytic methods, to arrive at the normal density. Then we construct the diffusion model to the central limit theorem introduced in the previous Section 7.3, Introduction to Statistics. This new section on diffusion grew out of a class taught at the University of Nevada for beginning graduate students in the earth sciences. The applications are to contaminant migration in the atmosphere and ground water. At the end of Chapter 8, Stochastic Models, is a new section on time series. This section also serves as an introduction to multivariate regression models with more than one predictor. As a natural follow-up to the discussion in Section 8.3, Linear Regression, the new section on time series introduces the important idea of correlation. It also shows how to recognize correlated variables and include the dependence structure in a time series model. The discussion is focused on autoregressive models, since these are the most generally useful time series models. They are also the most convenient, in that they can be handled using widely available linear regression software. For the benefit of students with access to a statistical package, this section illustrates the proper application and interpretation of advanced methods including autocorrelation plots and sequential sums of squares. However, the entire section can also be covered using only a basic implementation of regression that allows multiple predictors and outputs the two basic measures \( R^2 \) and the residual standard deviation \( s \). This can all be done with a good spreadsheet or hand calculator.

The third edition also reflects the evolution of technology. The text includes output from the newest version of the computer algebra systems Maple and Mathematica. Spreadsheet implementations of linear and integer programming solvers are introduced in Chapter 3, along with output from the popular linear programming package LINDO. The sections on linear regression and time series contain computer outputs from the popular statistical package Minitab. All of the computer graphics and all discussions of the appropriate use of technology have been updated.

Support for instructors is now stronger than ever. A complete and detailed solutions manual for instructors is available from the author or the publisher, for instructors who adopt the text for classroom use. Computer implementations of the algorithms used in the text can be downloaded for a variety of platforms, along with the computer files used to produce all the graphics and computer outputs included in the text. These downloads are all available at www.stt.ssu.edu/~sczub/ modeling.html.
The response to the first two editions of the text was gratifying. The best part of this job is interacting with students and instructors who use this book. Please feel free to contact me with any comments or suggestions.

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Part I
OPTIMIZATION MODELS
Chapter 1

ONE VARIABLE OPTIMIZATION

Problems in optimization are the most common applications of mathematics. Whatever the activity in which we are engaged, we want to maximize the good that we do and minimize the unfortunate consequences or costs. Business managers attempt to control variables in order to maximize profit or to achieve a desired goal for production and delivery at a minimum cost. Managers of renewable resources such as fisheries and forests try to control harvest rates in order to maximize long-term yield. Government agencies set standards to minimize the environmental costs of producing consumer goods. Computer system managers try to maximize throughput and minimize delays. Farmers space their plantings to maximize yield. Physicians regulate medications to minimize harmful side effects. What all of these applications and many more have in common is a particular mathematical structure. One or more variables can be controlled to produce the best outcome in some other variable, subject in most cases to a variety of practical constraints on the control variables. Optimization models are designed to determine the values of the control variables which lead to the optimal outcome, given the constraints of the problem.

We begin our discussion of optimization models at a place where most students will already have some practical experience. One-variable optimization problems, sometimes called maximum-minimum problems, are typically discussed in first-semester calculus. A wide variety of practical applications can be handled using just these techniques. The purpose of this chapter, aside from a review of these basic techniques, is to introduce the fundamentals of mathematical modeling in a familiar setting.

1.1 The Five-Step Method

In this section we outline a general procedure that can be used to solve problems using mathematical modeling. We will illustrate this procedure, called the five-
CHAPTER 1. ONE VARIABLE OPTIMIZATION

step method, by using it to solve a one-variable maximum-minimum problem typical of those encountered by most students in the first semester of calculus.

Example 1.1. A pig weighing 200 lbs gains 5 lbs per day and costs 45 cents a day to keep. The market price for pigs is 65 cents per pound, but is falling 1 cent per day. When should the pig be sold?

The mathematical modeling approach to problem solving consists of five steps:

1. Ask the question.
2. Select the modeling approach.
3. Formulate the model.
4. Solve the model.
5. Answer the question.

The first step is to ask a question. The question must be phrased in mathematical terms, and it often requires a good deal of work to do this. In the process we are required to make a number of assumptions or suppositions about the problem. We can always come back and make a better guess later on. Before we can ask a question in mathematical terms we need to define our terms. We can then be ready to ask a question. Write down in explicit mathematical terms all variables, constants, and inequalities; and other assumptions that are relevant.

Next we need to list our assumptions about the variables, constants, and inequalities; and other assumptions that are relevant.

In Example 1.1 the weight $w$ of the pig (in lbs), the number of days $t$ until the pig is sold, the cost $C$ of keeping the pig $t$ days (in dollars), the market price $p$ for pigs ($/lb), the revenue $R$ obtained when we sell the pig ($), and our objective here is to maximize profit $P (\text{dollars})$ as follows:

\[
\begin{align*}
P &= C - R \\
C &= 0.45t \\
R &= p \cdot w \\
P &= R - C \\
t &\geq 0
\end{align*}
\]

Objective: Maximize $P$

Figure 1.1: Results of step 1 of the pig problem.

Notice that we have included units as a check that our equation makes sense. The other assumptions inherent in our problem are as follows:

\[
\begin{align*}
(p \text{ dollars}) &= \frac{0.65 \text{ dollars}}{\text{lb}} - \frac{0.01 \text{ dollars}}{\text{lb} \cdot \text{day}} (t \text{ days}) \\
(C \text{ dollars}) &= 0.45t \\
(R \text{ dollars}) &= p \cdot w \\
P \text{ (dollars)} &= (R \text{ dollars}) - (C \text{ dollars})
\end{align*}
\]

We are also assuming that $t \geq 0$. Our objective in this problem is to maximize our net profit, $P$. Figure 1.1 summarizes the results of step 1; in a form convenient for later reference:

The three stages of step 1 (variables, assumptions, and objective) need not be completed in any particular order. For example, it is often useful to determine the objective early in step 1. In Example 1.1, it may not be readily apparent that $R$ and $C$ are variables until we have defined our objective, $P$, and we recall that $P = R - C$. One way to check that step 1 is complete is to see whether our objective $P$ relates all the way back to the variable $t$. The best general advice about step 1 is to do something. Start by writing down whatever is immediately apparent (e.g., some of the variables can be found simply by realizing over the problem and looking for nouns), and the rest of the pieces will probably fall into place.

Step 2 is to select the modeling approach. Now that we have a problem stated in mathematical language, we need to select a mathematical approach to use to get an answer. Many types of problems can be stated in a standard
CHAPTER 1. ONE VARIABLE OPTIMIZATION

form for which an effective general solution procedure exists. Most research in
applied mathematics consists of identifying these general categories of problems
literature in this area, and many new advances continue to be made. Few, if
literature to make a good selection for the modeling approach. In this book,
our example problem will be modeled as a one-variable optimization problem,
or maximum-minimum problem.

We outline the modeling approach we have selected. For complete details
we refer the reader to any introductory calculus textbook.

We are given a real-valued function \( y = f(x) \) defined on a subset
\( S \) of the real line. There is a theorem that states that if \( f \) attains its
maximum or minimum at an interior point \( x \in S \), then \( f'(x) = 0 \),
assuming that \( f \) is differentiable at \( x \). This allows us to rule out
any interior point \( x \in S \) at which \( f'(x) \neq 0 \) as a candidate for max-
imum. This procedure works well as long as there are not too many
exceptional points.

Step 3 is to formulate the model. We need to take the question exhibited in
step 1 and reformulate it in the standard form selected in step 2, so that we can
apply the standard general solution procedure. It is often convenient to change
variable names if we will refer to a modeling approach that has been described
using specific variable names, as is the case here. We write

\[
P = R - C = (p \cdot w - 0.45) - (0.65 - 0.01x)(200 + 5x) - 0.45x.
\]

Let \( y = P \) be the quantity we wish to maximize and \( x = t \) the independent
variable. Our problem now is to maximize

\[
y = f(x) = (0.65 - 0.01x)(200 + 5x) - 0.45x
\]

over the set \( S = \{x : x \geq 0\} \).

Step 4 is to solve the model, using the standard solution procedure identified
defined by Eq. (1.1) over the set \( x \geq 0 \). Figure 1.2 shows a graph of the function
\( f(x) \). Since \( f \) is quadratic in \( x \), the graph is a parabola. We compute that

\[
f'(x) = \frac{8 - 2x}{10},
\]

so that \( f'(x) = 0 \) at the point \( x = 8 \). Since \( f \) is increasing on the interval
\(( -\infty, 8) \) and decreasing on \( [8, \infty) \), the point \( x = 8 \) is the global maximum. At

Figure 1.2: Graph of net profit \( f(x) = (0.65 - 0.01x)(200 + 5x) - 0.45x \) versus
time to sell \( x \) pigs for the pig problem.

In this problem we have \( y = f(8) = 133.20 \). Since the point \( (x, y) = (8, 133.20) \) is
the global maximum of \( f \) over the entire real line, it is also the maximum over
the set \( x \geq 0 \).

Step 5 is to answer the question posed originally in step 1; i.e., when to sell
the pig in order to maximize profit. The answer obtained by our mathematical
model is to sell the pig after eight days, thus obtaining a net profit of $133.20.
This answer is valid only as long as the assumptions made in step 1 remain valid.
Related questions and alternative assumptions can be addressed by changing
what we did in step 1. Since we are dealing with a real problem (A farmer owns
pigs. When should they be sold?), there is an element of risk involved in step 1.
For that reason it is usually necessary to investigate several alternatives. This
process, called sensitivity analysis, will be discussed in the next section.

The basic purpose of this section was to introduce the five-step method for
mathematical modeling. Figure 1.3 summarizes the method in a form convenient
for later reference. In this book we will apply the five-step method to solve a
wide variety of problems in mathematical modeling. Our discussion of step 2
will generally include a description of the modeling approach selected, along
with an example or two. The reader who is already familiar with the modeling
approach may choose to skip this part, or just skim to pick up the notation.
Some of the other points summarized in Fig. 1.3, such as the use of "appropriate
technology," will be expanded upon later in this book.

The exercises at the end of each chapter also require the application of the
CHAPTER 1. ONE VARIABLE OPTIMIZATION

Step 1. Ask the question.
- Make a list of all the variables in the problem, including appropriate units.
- Be careful not to confuse variables and constants.
- State any assumptions you are making about these variables, including equations and inequalities.
- Check units to make sure that your assumptions make sense.
- State the objective of the problem in precise mathematical terms.

Step 2. Select the modeling approach.
- Choose a general solution procedure to be followed in solving this problem.
- Generally speaking, success in this step requires experience, skill, and familiarity with the relevant literature.
- In this book we will usually specify the modeling approach to be used.

Step 3. Formulate the model.
- Restate the question posed in step 1 in the terms of the modeling approach specified in step 2.
- You may need to relabel some of the variables specified in step 1 in order to agree with the notation used in step 2.
- Note any additional assumptions made in order to fit the problem described in step 1 into the mathematical structure specified in step 2.

Step 4. Solve the model.
- Apply the general solution procedure specified in step 2 to the specific problem formulated in step 3.
- Be careful in your mathematics. Check your work for math errors. Does your answer make sense?
- Use appropriate technology. Computer algebra systems, graphics, and numerical software will increase the range of problems within your grasp, and they also help reduce math errors.

Step 5. Answer the question.
- Rephrase the results of step 4 in nonmathematical terms.
- Avoid mathematical symbols and jargon.
- Anyone who can understand the statement of the question as it was presented to you should be able to understand your answer.

1.2 SENSITIVITY ANALYSIS

five-step method. Getting in the habit of using the five-step method now will make it easier to succeed on the more difficult modeling problems to come. Be sure to pay particular attention to step 5. In the real world, it is not enough to be right. You also need the ability to communicate your findings to others, some of whom may not be as mathematically knowledgeable as you.

1.2 Sensitivity Analysis

The preceding section outlines the five-step approach to mathematical modeling. The process begins by making some assumptions about the problem. We are rarely certain enough about things to be able to expect all of those assumptions to be exactly valid. Therefore, we need to consider how sensitive our conclusions are to each of the assumptions we have made. This kind of sensitivity analysis is an important aspect of mathematical modeling. The details vary according to the modeling approach used, and so our discussion of sensitivity analysis will continue throughout this book. We will focus here on sensitivity analysis for simple one-variable optimization problems.

In the preceding section we used the pig problem (Example 1.1) to illustrate the five-step approach to mathematical modeling. Figure 1.1 summarizes the assumptions we made in solving that problem. In this instance the data and assumptions were mostly spelled out for us. Even so, we need to be critical. Data are obtained by measurement, observation, and sometimes sheer guess. We need to consider the possibility that the data are not precise.

Some data are naturally known with much more certainty than others. The current weight of the pig, the current price for pigs, and the cost per day of keeping the pig are easy to measure and are known to a great degree of certainty. The rate of growth of the pig is a bit less certain, and the rate at which the price is falling is even less certain. Let \( r \) denote the rate at which the price is falling. We assumed that \( r = 0.03 \) dollars per day, but let us now suppose that the actual value of \( r \) is different. By repeating the solution procedure for several different values of \( r \), we can get an idea of the sensitivity of our answer to the value of \( r \). Table 1.1 shows the results of solving our problem for a few selected values of \( r \). Figure 1.4 contains the same sensitivity data in graphical form. We can see that the optimal time to sell is quite sensitive to the parameter \( r \).

A more systematic method for measuring this sensitivity would be to treat \( r \) as an unknown parameter, following the same steps as before. Writing

\[
p = 0.65 - rt,
\]

we can proceed as before to obtain

\[
y = f(x) = (0.65 - rx)(200 + 5x) - 0.45x.
\]

Then we can compute

\[
f'(x) = -2(25x + 50r) - 7
\]

5

we can proceed as before to obtain
1.2. SENSITIVITY ANALYSIS

<table>
<thead>
<tr>
<th>$r$ ($$/\text{day}$$)</th>
<th>$x$ (days)</th>
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<tr>
<td>0.008</td>
<td>15.0</td>
</tr>
<tr>
<td>0.009</td>
<td>11.1</td>
</tr>
<tr>
<td>0.010</td>
<td>8.0</td>
</tr>
<tr>
<td>0.011</td>
<td>5.5</td>
</tr>
<tr>
<td>0.012</td>
<td>3.3</td>
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Table 1.1: Sensitivity of best time to sell $x$ to rate $r$ at which price is falling for the pig problem.

Figure 1.5: Graph of net profit $f(x) = (0.65 - 0.015x)(200 + 5x) - 0.45x$ versus time to sell $x$ for the pig problem in the case $r = 0.015$.

so that $f'(x) = 0$ at the point

$$x = \frac{(7 - 500r)}{25r}.$$  \hspace{1cm} (1.2)

The optimal time to sell is given by Eq. (1.2) as long as this expression is positive, i.e., as long as $0 < r \leq 0.014$. For $r > 0.014$, the vertex of the parabola $y = f(x)$ lies outside of the set $x \geq 0$ over which we are maximizing. In this case the optimal time to sell is at $x = 0$ since we have $f' < 0$ on the entire interval $[0, \infty)$. See Figure 1.5 for an illustration in the case $r = 0.015$.

We are also uncertain about the growth rate $g$ of the pig. We have assumed that $g = 0$ (lb/day). More generally, we have that

$$w = 200 + gt,$$

which leads to the equation

$$f(x) = (0.65 - 0.015x)(200 + gx) - 0.45x,$$

so that

$$f'(x) = \frac{-2gx + 5(49 - 13g)}{100}.$$  \hspace{1cm} (1.3)

Now $f'(x) = 0$ at the point

$$x = \frac{5(13g - 49)}{2g}.$$
CHAPTER 1. ONE VARIABLE OPTIMIZATION

Figure 1.6: Graph of best time to sell \( x \) versus growth rate \( g \) for the pig problem.

The optimal time to sell is given by Eq. (1.3) so long as \( x \) represents a non-negative value of \( x \). Figure 1.6 shows the relationship between the growth rate \( g \) and the optimal time to sell.

It is most natural and most useful to interpret sensitivity data in terms of relative change or percent change, rather than in absolute terms. For example, a 10\% decrease in \( r \) leads to a 39\% increase in \( x \), while a 10\% decrease in \( g \) leads to a 34\% decrease in \( x \). If \( x \) changes by an amount \( \Delta x \), the relative change in \( x \) is given by \( \Delta x/x \), and the percent change in \( x \) is \( 100\Delta x/x \). If \( r \) changes by \( \Delta r \), resulting in the change \( \Delta x \) in \( x \), then the ratio between the relative changes is \( \Delta x/x \) divided by \( \Delta r/r \). Letting \( \Delta x \to 0 \) and using the definition of the derivative, we obtain

\[
\frac{\Delta x}{x} = \frac{dx}{dr} \frac{r}{x} \Delta r.
\]

We call this limiting quantity the sensitivity of \( x \) to \( r \), and we will denote it by \( S(x, r) \). In the pig problem we have

\[
\frac{dx}{dr} = \frac{-7}{2g^2} = \frac{-2,800}{8} \quad \text{at the point } r = 0.01 \text{ and } x = 8;
\]

so that a 1\% increase in the growth rate of the pig would cause us to wait about 3\% longer to sell the pig.

If \( r \) goes up by 2\%, then \( x \) goes down by 7\%. Since

\[
\frac{dx}{dg} = \frac{245}{2g^2} = 4.9,
\]

we have

\[
S(x, g) = \frac{dx}{dg} \frac{g}{x} = (4.9) \left( \frac{2}{8} \right) = 3.3625,
\]

so that a 25\% error in \( g \) would be quite surprising if we have observed the past history of growth in this pig or in similar animals. A 25\% error in our estimate of \( r \) would not be at all surprising.

1.3 Sensitivity and Robustness

A mathematical model is robust if the conclusions it leads to remain true even though the model is not completely accurate. In real problems we will never have perfect information, and even if it were possible to construct a perfectly accurate model, we might be better off with a simpler and more tractable approximation. For this reason a consideration of robustness is a necessary ingredient in any mathematical modeling project.

In the preceding section we introduced the process of sensitivity analysis, which is a way to gauge the robustness of a model with respect to assumptions.
CHAPTER 1. ONE VARIABLE OPTIMIZATION

about the data. There are other assumptions made in step 1 of the mathematical modeling process which should also be examined. While it is often necessary to be realistic in the modeling to see if it is so specialized as to invalidate the results of the modeling process.

Figure 1.3 contains a summary of the assumptions made in solving the pig problem. In this case the assumptions are both the weight and the selling price per pound are linear functions of time. These assumptions are not expected to hold exactly. After all, according to these assumptions, a year from now the pig will weigh

\[ w = 200 + 5(365) \]

2,025 lbs

and sell for

\[ p = 0.65 - 0.01(365) \]

\[-3.00\text{ dollars/lb.}\]

A more realistic model would take into account both the nonlinearity of these functions and the increasing uncertainty as time goes on.

How can a model give the right answer if the assumptions are wrong? While mathematical modeling strives for perfection, perfection can never be achieved. It would be more descriptive to say that mathematical modeling strives toward perfection. A well-constructed mathematical model will be robust, which is to say that while the answers it gives may not be perfectly correct, they will be close enough to be useful in a real-world context.

Let us examine the linear assumptions made in the pig problem. Our basic equation is

\[ P = \text{pw} - 0.05t \]

where \( P \) is the selling price of the pig in dollars per pound, and \( w \) is the weight of the pig in pounds. If the original data and assumptions of the model are not too far off, then the best time to sell the pig is obtained by setting \( P' = 0 \). Calculate to find

\[ p'w = p'w_0 = 0.45 \]

dollars per day. The term \( p'w = p'w_0 \) represents the rate of increase in the value of the pig. Our model tells us to keep the pig as long as the value of the pig in value has two components, \( p'w \) and \( p'w_0 \). The first term, \( p'w \), represents the decrease in value due to a drop in price. The second term, \( p'w_0 \), represents the loss in the application of this general model. The data required include a complete

1.4 EXERCISES

1. An automobile manufacturer makes a profit of $1,500 on the sale of a certain model. It is estimated that for every $100 of rebate, sales increase by 15%.

(a) What amount of rebate will maximize profit? Use the five-step method, and model as a one-variable optimization problem.

(b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit.

(c) Suppose that rebates actually generate only a 10% increase in sales per $100. What is the effect? What if the response is somewhere between 10 and 15% per$100 of rebate?

(d) Under what circumstances would a rebate offer cause a reduction in profit?

2. In the pig problem, perform a sensitivity analysis based on the cost per day of keeping the pig. Consider both the cost at the time to sell and on the resulting profit. If a new feed costing 60 cents/day would let us construct a realistic scenario. The farmer has a pig weighing approximately 200 lbs. The pig has been growing about five lbs/day during the last week. Five days ago the pig could have been sold for 70 cents/lb but by now the price has dropped to 65 cents/lb. What should we do? The obvious approach is to project on the basis of this data (\( w = 200, w' = 5, p = 0.65, p' = -0.01 \) and decide when to sell. This is exactly what we did. We understand that \( p' \) and \( w' \) will not remain constant over the next few weeks, and that therefore \( p \) and \( w \) will not be linear functions of time. However, so long as \( p' \) and \( w' \) do not change too much over this period, the error involved in assuming they remain constant will not be too great. We are now prepared to give a somewhat broader interpretation to the results of our sensitivity analysis from the preceding section. Recall that the sensitivity of the best time to sell (\( z \)) to changes in the growth rate \( w' \) was calculated to be

3. Suppose that in fact the growth rate over the next few weeks is somewhere between 4.5 and 5.5 lbs/day. This is within 10% of the assumed value. Then the best time to sell the pig will be within 30% of 8 days, or between 5 and 11 days. The amount of lost profit by selling at 8 days is less than 1 dollar.

With regard to price, suppose that we feel the value \( p' = -0.01 \), or a 1 cent/day drop in price over the next few weeks, is a worst-case scenario. Prices are likely to drop more slowly in the future and may even level off (\( p' = 0 \)). All we can really say now is that we should wait at least 8 days to sell. For small values of \( p' \) (near zero), our model suggests waiting a very long time to sell. However, our model is not valid over long time intervals. The best course of action in this case is probably to keep the pig for a week, recompute the parameter values \( p', p', w', \text{ and } w, \) and start over.
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the pig grow at a rate of 7 lbs/day, would it be worth switching feed? What is the minimum improvement in growth rate that would make this new feed worthwhile?

3. Reconsider the pig problem of Example 1.1, but now assume that the price for pigs is to level off. Let

$$p = 0.65 - 0.014 + 0.000005t$$

represent the price for pigs (cents/lb) after t days.

(a) Graph Eq. (1.4) along with our original price equation. Explain why our original price equation could be considered as an approximation to Eq. (1.4) for values of t near zero.

(b) Find the best time to sell the pig. Use the five-step method, and model as a one-variable optimization problem.

(c) The parameter 0.000005 represents the rate at which price is leveling off. Conduct a sensitivity analysis on this parameter. Consider both the optimal time to sell and the resulting profit.

(d) Compare the results of part (b) to the optimal solution contained in the text. Comment on the robustness of our assumptions about price.

4. An oil spill has fouled 200 miles of Pacific shoreline. The oil company responsible has been given 14 days to clean up the shoreline, after which a fine will be levied in the amount of $10,000/day. The local cleanup crew can scrub five miles of beach per week at a cost of $500/day. Additional crews can be brought in at a cost of $13,000 plus $800/day for each crew.

(a) How many additional crews should be brought in to minimize the total cost to the company? Use the five-step method. How much will the clean-up cost?

(b) Examine the sensitivity to the rate at which a crew can clean up the shoreline. Consider both the optimal number of crews and the total cost to the company.

(c) Examine the sensitivity to the amount of the fine. Consider the number of days the company will take to clean up the spill and the total cost to the company.

(d) The company has filed an appeal on the grounds that the amount of the fine is excessive. Assuming that the only purpose of the fine is to motivate the company to clean up the oil spill, does the fine seem excessive?

5. It is estimated that the growth rate of the fin whale population (per year) is

$$r = (1 - x/K)$$

where $$r > 0.08$$ is the intrinsic growth rate, $$K = 400,000$$ is the maximum sustainable population, and $$x$$ is the current population.

now around 70,000. It is further estimated that the number of whales harvested per year is about 0.000001 Ex, where $$E$$ is the level of fishing effort in boat-days. Given a fixed level of effort, population will eventually stabilize at the level where growth rate equals harvest rate.

(a) What level of effort will maximize the sustained harvest rate? Model as a one-variable optimization problem using the five-step method.

(b) Examine the sensitivity to the intrinsic growth rate. Consider both the optimum level of effort and the resulting population level.

(c) Examine the sensitivity to the maximum sustainable population. Consider both the optimum level of effort and the resulting population level.

6. In Exercise 5, suppose that the cost of whaling is $500 per boat-day, and the price of a fin whale carcass is $8,000.

(a) Find the level of effort that will maximize profit over the long term. Model as a one-variable optimization problem using the five-step method.

(b) Examine the sensitivity to the cost of whaling. Consider both the optimal profit in $/year and the level of effort.

(c) Examine the sensitivity to the price of a fin whale carcass. Consider both profit and level of effort.

(d) Over the past 30 years there have been several unsuccessful attempts to ban whaling worldwide. Examine the economic incentives for whalers to continue harvesting. In particular, determine the conditions (values of the two parameters: cost per boat-day and price per fin whale carcass) under which harvesting the fin whale produces a sustained profit over the long term.

7. Reconsider the pig problem of Example 1.1, but now suppose that our objective is to maximize our profit rate ($/day). Assume that we have already owned the pig for 90 days and have invested $100 in this pig to date.

(a) Find the best time to sell the pig. Use the five-step method, and model as a one-variable optimization problem.

(b) Examine the sensitivity to the growth rate of the pig. Consider both the best time to sell and the resulting profit rate.

(c) Examine the sensitivity to the rate at which the price for pigs is dropping. Consider both the best time to sell and the resulting profit rate.

8. Reconsider the pig problem of Example 1.1, but now take into account the fact that the growth rate of the pig decreases as the pig gets older. Assume that the pig will be fully grown in another five months.
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(a) Find the best time to sell the pig in order to maximize profit. Use the five-step method, and model as a one-variable optimization problem.
(b) Examine the sensitivity to the time it will take until the pig is fully grown. Consider both the best time to sell and the resulting profit.

9. A local daily newspaper with a circulation of 80,000 subscribers is thinking of raising its subscription price. Currently the price is $1.50 per week, and it is estimated that the paper would lose 5,000 subscribers if the rate were to be raised by ten cents/week.

(a) Find the subscription price that maximizes profit. Use the five-step method, and model as a one-variable optimization problem.
(b) Examine the sensitivity of your answer in part (a) to the assumption of 5,000 lost subscribers. Calculate the optimal subscription rate assuming that this parameter is 3,000, 4,000, 5,000, 6,000, or 7,000.
(c) Let $n = 5,000$ denote the number of subscribers lost when the subscription price increases by ten cents. Calculate the optimal subscription price $p$ as a function of $n$, and use this formula to determine the sensitivity $S(p, n)$.
(d) Should the paper change its subscription price? Justify your conclusions in plain English.

Further Reading


Chapter 2
MULTIVARIABLE OPTIMIZATION

Many optimization problems require the simultaneous consideration of a number of independent variables. In this chapter we consider the simplest category of multivariable optimization problems. The techniques should be familiar to most students from multivariable calculus. In this chapter we also introduce the use of computer algebra systems to handle some of the more complicated algebraic computations.

2.1 Unconstrained Optimization

The simplest type of multivariable optimization problems involves finding the maximum or minimum of a differentiable function of several variables over a nice set. Further complications arise, as we will see later, when the set over which we optimize takes a more complex form.

Example 2.1. A manufacturer of color TV sets is planning the introduction of two new products, a 19-inch LCD flat panel set with a manufacturer's suggested retail price (MSRP) of $339 and a 21-inch LCD flat panel set with an MSRP of $399. The cost to the company is $195 per 19-inch set and $225 per 21-inch set, plus an additional $480,000 in fixed costs. In the competitive market in which these sets will be sold, the number of sales per year will affect the average selling price. It is estimated that for each type of set, the average selling price drops by one cent for each additional unit sold. Furthermore, sales of the 19-inch set will affect sales of the 21-inch set, and vice-versa. It is estimated that the average selling price for the 19-inch set will be reduced by an additional 0.3 cents for each 21-inch set sold, and the price for the 21-inch set will decrease by 0.4 cents for each 19-inch set sold. How many units of each type of set should be manufactured?