Module 792

The Spread of Forest Fires

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Application Field: Biology, Forestry
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TITLE: The Spread of Forest Fires

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MATHEMATICAL FIELD: Probability, Calculus

APPLICATION FIELD: Biology, Forestry

TARGET AUDIENCE: Students with a background in infinite series, e.g., from second-semester calculus.

ABSTRACT: We create a simple discrete probabilistic model for spread of a forest fire. We examine the conditions for which the fire will either die out or spread indefinitely, identifying and bounding a critical value for the probability of transmission of the fire to an immediately adjacent location.

PREREQUISITES: Summation of geometric series, elementary probability, intuitive limits; for the Appendix, derivatives and limits of rational functions.
# The Spread of Forest Fires

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The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

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Paul J. Campbell
Solomon Garfunkel

Editor
Executive Director, COMAP
1. Introduction

- “What does research in mathematics look like?”
- “What do mathematicians do, exactly?”
- “Is there any thing new going on in mathematics?”

Such questions are exciting, because students asking them are curious about mathematics, and frustrating, because the students often do not have enough background in mathematics to tackle a research problem. The challenge to the instructor is to respond with an easily understood research question whose solution is within the grasp of a freshman or sophomore. An added bonus would be to convey the creative mix of ideas that is often crucial in a solution.

The project that follows is one possible response. It begins with a very approachable model for the spread of a forest fire. The question of interest is whether the fire will die out on its own, and this question leads to a hunt for a threshold value for the probability of further transmission of the fire. With use of the formula for the sum of a geometric series plus a minimal amount of probability, a partial answer is attainable: some bounds on the critical probability of spread of the fire.

The main ideas in this Module have been used

- in senior comprehensive projects,
- as a basis for talks given to undergraduates with a second-semester calculus background, and
- as an independent study for a motivated sophomore.

As organized here, the Module is written as an independent project for a student or a group of students. The time frame for the project is 2–4 weeks.

2. Forest Fire Model

The model is constructed on the $\mathbb{Z}^2$ lattice, where each lattice point $X = (x, y), x, y \in \mathbb{Z}$, is the site of a tree. The tree at the origin is struck by lightning and set ablaze. From any tree, the fire can spread in one step to any one of its four nearest neighbors, i.e., to any of the trees at $(1, 0), (0, 1), (-1, 0)$ or $(0, -1)$, independent of one another, each with transmission probability $p, 0 \leq p \leq 1$.

We keep track of the spread of the fire using a discrete time clock. Thus, at time $n = 0$, the tree at the origin is on fire and all other trees in the forest are unburnt. At time $n = 1$, the fire may have spread to one or more nearest neighbors of the tree at the origin. Also at time $n = 1$, the tree at the origin is completely burnt and therefore cannot be set on fire again nor transmit fire further.
We formalize the fire spread by assigning to each lattice site $X$ a state at time $n$, $n = 0, 1, 2, \ldots$ The three possible states for a tree are $U$ (unburnt), $F$ (on fire), and $B$ (burnt). The function $S_n(X)$ gives the state of the tree at site $X$ at time $n$.

Exercise

1. The spread of the fire is determined by the following rules. Translate them into words; for example, $S_n(X) = B$ translates to “the tree at site $X$ at time $n$ is burnt.”
   
   a) If $S_n(X) = B$, then $S_{n+1}(X) = B$.
   
   b) If $S_n(X) = F$, then $S_{n+1}(X) = B$.
   
   c) If $S_n(X) = U$, then $S_{n+1}(X)$ can be either $U$ or $F$, depending on the state of the neighboring trees and the transmission of the fire.

Case c) requires some exploration. A burning tree spreads the fire to neighboring trees independently of other burning trees. If events $A$ and $B$ are independent, then the probability that both events $A$ and $B$ occur, $P(A \cap B)$, is equal to $P(A) \cdot P(B)$, the product of the probabilities of each separate event. In other words, the proportion of the time that event $A$ occurs is the same when event $B$ occurs as when $B$ does not occur.

With this precise definition of independence, consider the unburnt tree at site $X$ with neighboring trees at $X_1$, $X_2$, $X_3$, and $X_4$, as in Figure 1.

![Figure 1](image_url)  

Figure 1. An unburnt tree at $X$ with neighboring trees.

Exercises

2. If $S_n(X_1) = F$, $S_n(X_2) = F$, $S_n(X_3) = B$, and $S_n(X_4) = U$, the tree at $X$ will remain unburnt at time $n + 1$ only if the fire at sites $X_1$ and $X_2$ do not spread to $X$. The probability of not spreading the fire from a burning tree to a neighbor is $1 - p$.

Let $A$ be the event that site $X_1$’s fire does not spread to site $X$, and let $B$ be the event that site $X_2$’s fire does not spread to site $X$. Using the
independence of the fire spread from site to site, the probability that site \( X \) remains in state \( U \) at time \( n + 1 \) is

\[
P(A \cap B) = P(A) \cdot P(B) = (1-p)^2.
\]

What is the probability that the site \( X \) switches to state \( F \) at time \( n + 1 \)? That is, what is the probability that the tree at \( X \) is on fire at time \( n + 1 \)?

3. Consider this assignment of states to sites: \( S_n(X_1) = F \), \( S_n(X_2) = B \), \( S_n(X_3) = F \), and \( S_n(X_4) = F \).
   a) If \( S_n(X) = U \), what is the probability that \( S_{n+1}(X) = U \)?
   b) Again, if \( S_n(X) = U \), what is the probability that \( S_{n+1}(X) = F \)?

4. Consider the case where at time \( n \) exactly \( i \) neighboring trees are on fire, \( i = 0, 1, \ldots, 4 \).
   a) If \( S_n(X) = U \), what is the probability that \( S_{n+1}(X) = U \)?
   b) Again, if \( S_n(X) = U \), what is the probability that \( S_{n+1}(X) = F \)?

### 3. Spread of the Forest Fire

The spread of the fire depends on the value of the transmission probability \( p \). If \( p \) is small, the fire will die out eventually. If \( p \) is large, then there is some chance that the fire will spread “infinitely,” that is, arbitrarily far from the initial lightning strike.

**The Big Question:** How large does \( p \) have to be for there to be a chance for indefinite spread of the fire?

Essentially, we are hunting for a **threshold value** or **critical value** \( p_c \). For \( p \in [0, p_c) \), the forest fire will die out. For \( p \in (p_c, 1] \), the fire has a chance of infinite spread. However, unless \( p = 1 \), it is not certain that there will be infinite spread of the fire.

The model plays out differently depending on the value of \( p \); the value \( p_c \) marks a **phase transition** in the model. Phase transitions abound in natural phenomena, too, as in the change of water from a liquid state to a solid state, which (for pure water at sea level) occurs at a critical temperature \( t_c = 0^\circ \text{C} \).

**Exercise**

5. Computer simulation can provide some insight into the critical value \( p_c \).
   Write a computer program that starts with one tree burning at the center of a \( 100 \times 100 \) lattice and shows the states of the trees at time \( n \) using the rules for spread established in the previous section. Explore the fire spread with different values of the transmission probability \( p \); to begin, test \( p = 0.2 \), \( p = 0.4 \), \( p = 0.6 \), and \( p = 0.8 \). Look at the states of the entire lattice at
various times, for example, at \( n = 20, n = 40, \) and \( n = 60, \) to see how the fire is spreading over time. For each value of \( p, \) run the program several times to observe any trends in how the fire spreads. You may also want to increase the size of the lattice to explore further. What do you notice about the spread of the fire? For which values of \( p \) does the fire appear to die out? For which values does the fire seem to continue spreading?

For convenience, let \( E_\infty \) be the event that the fire spreads infinitely. Hence for \( p < p_c, \) the probability of \( E_\infty \) is 0, while if \( p > p_c, \) the probability of \( E_\infty \) is greater than 0. The critical probability is therefore defined to be

\[
p_c = \inf \{ p \mid P(E_\infty) > 0 \}
\]

where \( \inf \) denotes the greatest lower bound of the set.

4. Lower Bound for \( p_c \)

To study infinite spread of the fire, we start by considering how the fire spreads from the origin to another site at a finite distance away. As the fire spreads across the lattice, it creates burnt paths through the forest. We consider one such path.

Exercises

6. a) Plot all of the \( \mathbb{Z}^2 \) lattice points \((x, y)\) where \(-4 \leq x, y \leq 4.\) At time \( n = 0, \) the tree at the origin is on fire. Suppose that the fire spreads to one of the neighboring trees. Pick one neighboring tree, at site \( X, \) say. How many options are there for choice of \( X? \) On your plot, draw the first step of the fire’s path from the origin to \( X.\)

b) For the next step in the spread of the fire, pick a tree neighboring \( X; \) call the site \( Y. \) How many options are there for choice of \( Y? \) On your plot, draw the second step of the fire’s spread.

c) For the third step, pick a tree neighboring \( Y. \) How many options are there? On your plot, draw the next step of the fire’s spread.

Because the fire cannot revisit a burnt tree, it maps out a self-avoiding path, meaning that the path cannot return to a site it has already visited.

Exercise

7. a) How many self-avoiding paths with a length of 2 steps are there on this lattice? How many with three steps?

b) Are there \( 4 \cdot 3 \cdot 3 \cdot 3 \) self-avoiding paths with a length of 4 steps? Sketch several paths and think about the question.
Depending on where the fire is at time \( n = 3 \), it might not have three options of sites to move to at time \( n = 4 \). For example, suppose that the first step from the origin is east, the second step is north, and the third step is west. Then the fourth step must be either north or west.

Counting the number of self-avoiding paths with \( n \) steps is a difficult problem, but we can at least establish an upper bound on the number of such paths:

\[
\text{Number of self-avoiding paths of } n \text{ steps } \leq 4 \cdot 3^{n-1}.
\]

Then the probability that the fire spreads along a self-avoiding path of \( n \) steps is

\[
\frac{\text{Number of } n\text{-step self-avoiding paths}}{\text{Number of self-avoiding paths of } n \text{ steps}} \times \text{Probability of fire spread } n \text{ times}.
\]

\[
= \frac{\text{Number of self-avoiding paths of } n \text{ steps} \times p^n}{(4 \cdot 3^{n-1}) p^n}.
\]

Now consider infinite spread of the fire; fire must spread along a self-avoiding path of infinite length. So let us consider fire spreading along a self-avoiding path with a large length.

Let \( Q \) be the probability of fire spreading along a self-avoiding path with \( N \) or more steps, where \( N \) is large. Then

\[
Q = P(\text{fire spreading along a self-avoiding path of length } \geq N)
\]

\[
\leq \sum_{n=N}^{\infty} P(\text{fire spreading along a self-avoiding path of length } n)
\]

\[
\leq \sum_{n=N}^{\infty} (4 \cdot 3^{n-1}) p^n
\]

\[
= 4p \sum_{n=N}^{\infty} (3p)^{n-1}.
\]

Exercise

8. If \( p < \frac{1}{3} \), then \( 0 \leq 3p < 1 \), and the sum in \( Q \) is the tail of a convergent geometric series. Use the closed form of the series to simplify the upper bound of \( Q \). Hint: The geometric series \( \sum_{n=0}^{\infty} ax^n \) converges to \( a/(1 - x) \) for \(-1 < x < 1 \) and \( a \in \mathbb{R} \). For more assistance, see the Appendix.

For infinite spread of the forest fire, we are interested in the limiting probability of fire spreading along a self-avoiding path of length \( N \), that is, the limit of \( Q \) as \( N \to \infty \).

Exercises

9. Assume that \( p < \frac{1}{3} \). Use the simplified upper bound of \( Q \) to find an upper bound for \( \lim_{N \to \infty} Q \).
10. Note that

\[ P(E_\infty) = \lim_{N \to \infty} Q, \]

so we can conclude that if \( p < \frac{1}{3} \), then the probability of infinite fire spread is 0. What then can be said about the value of the critical probability \( p_c \)?

5. Upper Bound for \( p_c \)

To obtain an upper bound for \( p_c \), we consider the spread of a fire at time \( n = 0 \) from a set \( I \) of \( M \) initial adjacent sites in a row: \((0, 0), (1, 0), (2, 0), \ldots, (M, 0)\). The fire spreads along self-avoiding paths; so let \( J \) be the set of sites \( X \) such that there is a self-avoiding path from a site in \( I \) to \( X \). That is, \( J \) consists of all sites that are burned by the fire’s spread; note that \( I \subset J \). See Figure 2 for an illustration.

![Figure 2](image)

Figure 2. Illustration of a fire starting from a row of trees from 0 to \( M \).

For each site \( Y \) in \( J \), construct a closed unit square centered at \( Y \). So if \( Y = (x, y) \), then the unit square has vertices at \((x \pm \frac{1}{2}, y \pm \frac{1}{2})\). Let \( A \) be the region of the plane covered by these squares, and let \( A \) have boundary \( B \). See Figure 3 for an illustration.

If there is indeed a boundary around all the sites that are burnt, then the fire did not spread infinitely from the initial sites. So our attention turns to the probability of the existence of such a boundary \( B \).

In constructing the unit squares around sites in \( J \), we have effectively created a dual lattice \( W^2 \), where

\[ W^2 = \{(u, v) : u = x + \frac{1}{2}, \ v = y + \frac{1}{2} \ \text{for all} \ (x, y) \in \mathbb{Z}^2 \}. \]
The boundary $B$ connects lattice points in this dual lattice.

The spread of the fire in the $\mathbb{Z}^2$ lattice induces a configuration in the $\mathbb{W}^2$ lattice in the following way. In the $\mathbb{Z}^2$ lattice, let $X$ and $Y$ be neighboring sites. We call the segment of the lattice between the sites an \textit{edge}. If the fire has spread from one site to a neighboring site, we say the edge is \textit{open}; otherwise, the edge is \textit{closed}. Thus, an edge in the $\mathbb{Z}^2$ lattice is open with probability $p$, and as the model runs its course, each edge in $\mathbb{Z}^2$ is classified as either open or closed.

Every edge in the $\mathbb{Z}^2$ lattice crosses an edge in the $\mathbb{W}^2$ lattice. If the edge is open in $\mathbb{Z}^2$, the corresponding edge is declared to be open in the $\mathbb{W}^2$ lattice. In our example (Figure 3), the edge in $\mathbb{Z}^2$ from the origin to the point $(0, 1)$ is open because the fire has spread from the origin to this site. Therefore, the edge in $\mathbb{W}^2$ from $(-\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2})$ is also open. Thus a configuration in $\mathbb{W}^2$ of open and closed edges is determined by the configuration of open and closed edges in $\mathbb{Z}^2$.

The boundary $B$ consists of closed edges in $\mathbb{W}^2$. Furthermore, $B$ is a self-avoiding path until it closes the loop around the burnt area in its last step.

\textbf{Exercises}

11. What is the smallest possible number $N$ of edges in $B$? Consider the minimum size of the set $J$, and determine the minimum number of edges in $\mathbb{W}^2$ that would be required to bound $J$.

12. Consider a boundary $B$ of some length $k$, where $k \geq N$; it must be a path formed of $k$ closed edges. What is the probability that all $k$ edges are closed?
How many boundaries of length $k$ exist? Since a boundary is a self-avoiding path, the question is difficult to answer precisely. However, we can establish an upper bound for the number of boundaries of length $k$. Since the initial set $I$ is on the horizontal axis of $Z^2$, a boundary of length $k$ must contain a vertical edge between $W^2$ lattice points of the form $(j + \frac{1}{2}, -\frac{1}{2})$ and $(j + \frac{1}{2}, \frac{1}{2})$ for some integer $j$ where $0 \leq j < k$. Thus, there are at most $k$ options for an edge of this form; and moving in a self-avoiding fashion from that edge onward, there are at most $3^{k-1}$ options for the following $k - 1$ steps. Therefore, the number of boundaries of length $k$ is less than or equal to $k \cdot 3^{k-1}$.

Let $R$ be the probability of the existence of a closed boundary $B$ around set $J$. Then we have

$$R = P(\text{closed boundary } B \text{ exists})$$

$$= \sum_{k=N}^{\infty} P(\text{closed boundary } B \text{ of length } k \text{ exists})$$

$$\leq \sum_{k=N}^{\infty} (k \cdot 3^{k-1})(1 - p)^k$$

$$= (1 - p) \sum_{k=N}^{\infty} k[3(1 - p)]^{k-1},$$

where $N = 2M + 4$ is the minimum number of edges in the boundary.

**Exercise**

13. If $p > \frac{2}{3}$, then $0 \leq 3(1 - p) < 1$, and the sum is the tail of a convergent series. (The ratio test shows that the series $\sum_{k=1}^{\infty} k[3(1 - p)]^{k-1}$ is convergent, and Exercise A2 calculates what it converges to; but here we are concerned just with the tail of the series, the sum from the $N$th term on.) What happens to $R$ if $N$ is increased? Hint: See Exercise A2 in the Appendix.

Assume that $p > \frac{2}{3}$. If $N = 2M + 4$ is large enough, then $R < 1$. But if the probability of the existence of a closed boundary $B$ around set $J$ is less than 1, then the probability of no closed boundary around set $J$ is greater than 0. That implies that at least one of the trees in the initial set $I$ was the starting point of an infinite open self-avoiding path. Since it is equally likely for any one of these trees to be the starting point of such a path, we have
0 < P(\text{closed boundary } B \text{ does not exist})
= P(\text{set } I \text{ generates infinite fire spread})
\leq P(\text{infinite open self-avoiding path starting at } (0,0) \text{ exists})
+ P(\text{infinite open self-avoiding path starting at } (1,0) \text{ exists})
+ \ldots
+ P(\text{infinite open self-avoiding path starting at } (M,0) \text{ exists})
= (M + 1) P(\text{infinite open self-avoiding path starting at } (0,0) \text{ exists})
= (M + 1) P(E_{\infty}).

Exercise

14. So we can conclude that if \( p > \frac{2}{3} \), then the probability of infinite fire spread \( P(E_{\infty}) \) is greater than 0. What then can be said about the value of the critical probability \( p_c \)?

6. Conclusions

We now have bounds on the critical probability \( p_c \):

\[
\frac{1}{3} \leq p_c \leq \frac{2}{3}.
\]

You should review your computer simulation results to see how those results agree. The next mathematical step would be to improve the techniques so as to narrow the bounds of \( p_c \), and ultimately find the exact value of the critical probability. The proof that

\[
p_c \text{ actually equals } \frac{1}{2}
\]

was first published by Kesten [1980], and other proofs have subsequently appeared (e.g., Grimmett [1989]).

7. Areas for Further Exploration

There are other interesting questions besides infinite spread. For example, for transmission probabilities below the critical probability, we know that the fire will die out.

- How far will it spread on average before it dies out?
- What can be determined about the shape of the burnt region? Is it circular, or does it exhibit some other pattern?
The answers depend on the size of the transmission probability $p$.

We could explore variations on the model. The lattice for the model is square, with each tree having four nearest neighbors. However, we could also model using a lattice that is triangular or hexagonal, with each tree having three or six nearest neighbors. How can the methods for finding upper and lower bounds of $p_c$ be adjusted for a different lattice? Do the bounds change?

In the model, we held the transmission probability $p$ fixed. However, it is reasonable to consider that the probability of spread may vary. In particular, a constant wind would make it more likely for a fire to spread in one direction. Consider a constant wind from the south that makes the probability of fire spreading to the north more likely than toward any other direction. Use computer simulations to explore this with the probability of northward spread above the threshold value $p_c = \frac{1}{2}$ and the probability of eastern, western and southern spread below the threshold value. What do you predict will happen?

Another adjustment is to consider variable winds. A wind coming from the south with variable speed can be worked into the computer model as a random choice from several transmission probability values; the stronger the wind, the higher the probability of spread to the north. One could make it more complicated by having random winds from any direction at any time, or stronger winds as the fire grows.

There are many other models with similar structures and questions. For example, the model for the spread of forest fire can also be used to study the spread through a population of an immunity-granting illness such as measles or chicken pox (e.g., Durrett [1988a]). A closely related model is the percolation model. The model is set on the $\mathbb{Z}^2$ lattice with edges between sites open with probability $p$, and a central question is whether the origin is in an infinite open cluster. That is, is there is an open path for fluid (such as water or oil) to move from the “origin” to “infinity”?

This project is an entry point into the study of interacting particle systems and cellular automata. Those models all follow simple rules to determine the states of sites, and yet they produce complex behaviors for study. Consider the most famous cellular automaton of all, John Conway’s Game of Life. In this model, sites have only two states, alive and dead, and at each time step, sites change states depending on the states of their nearest neighbors. Even with these simplest of transition rules, interesting evolutions occur (see Gardner [1970; 1983] and computer implementations such as Schaffhauser [1996]).
8. Solutions to Selected Exercises

1. a) If the tree at site $X$ at time $n$ is burnt, then the tree at that site is burnt at the following time $n + 1$.
   b) If the tree at site $X$ at time $n$ is on fire, then the tree at that site is burnt at the following time $n + 1$.
   c) If the tree at site $X$ at time $n$ is unburnt, then the tree at that site is either unburnt or on fire at the following time $n + 1$.

2. The probability that the tree at site $X$ is on fire at time $n + 1$ is $1 - (1 - p)^2$.

3. a) If $S_n(X) = U$, the probability that $S_{n+1}(X) = U$ is $(1 - p)^3$.
   b) If $S_n(X) = U$, the probability that $S_{n+1}(X) = U$ is $1 - (1 - p)^3$.

4. a) If $S_n(X) = U$, the probability that $S_{n+1}(X) = U$ is $(1 - p)^i$.
   b) If $S_n(X) = U$, the probability that $S_{n+1}(X) = U$ is $1 - (1 - p)^i$.

6. a) There are 4 options for the choice of $X$.
   b) There are 3 options for the choice of $Y$.
   c) There are 3 options for the third step.

7. There are $4 \cdot 3 = 12$ self-avoiding paths of length 2 and $4 \cdot 3 \cdot 3 = 36$ self-avoiding paths of length 3.
   a) There are fewer than $4 \cdot 3 \cdot 3 \cdot 3$ self-avoiding paths of length 4.

8. Let $x = 3p$. Then $Q \leq 4p \sum_{n=N}^{\infty} (3p)^{n-1} = 4p \frac{(3p)^{N-1}}{1 - 3p}$.

9. $\lim_{N \to \infty} Q \leq 4p \lim_{N \to \infty} \frac{(3p)^{N-1}}{1 - 3p} = 0$.

10. Since $p_c = \inf \{ p : P(E_\infty) > 0 \}$, $p_c$ cannot be less than $\frac{1}{3}$. Thus $p_c \geq \frac{1}{3}$.

11. The minimum set is $I$ itself, and for this set, $N = 2(M + 1) + 2 = 2M + 4$.

12. The probability of $k$ edges being closed is $(1 - p)^k$ by independence.

13. If $N$ is increased, the probability $R$ decreases.

14. Since $p_c = \inf \{ p : P(E_\infty) > 0 \}$, $p_c$ cannot be greater than $\frac{2}{3}$. Thus $p_c \leq \frac{2}{3}$. 

9. Appendix

We give more details on the geometric series used in several exercises. Let $x$ be a real number. For some integer $M$, define the sum $S_M$ to be

$$S_M = 1 + x + x^2 + x^3 + \cdots + x^M = \sum_{k=0}^{M} x^k.$$

Exercises
A1. Consider the derivative of $S_M$:

$$\frac{dS_M}{dx} = 0 + 1 + 2x + 3x^2 + \cdots + Mx^{M-1} = \sum_{n=1}^{M} n x^{n-1}.$$  

Take the derivative of the closed form of $S_M$ to find another expression of the derivative:

$$\frac{dS_M}{dx} = \frac{d}{dx} \left( \frac{1 - x^{M+1}}{1 - x} \right) = \cdots.$$  

Let $S' = \lim_{M \to \infty} \frac{dS_M}{dx}$. Does this limit exist for $-1 < x < 1$?

A2. For $-1 < x < 1$, let $N = M + 1$ and define the tail of the series

$$T_N = S' - \frac{dS_M}{dx} = \sum_{n=N}^{\infty} n x^{n-1}.$$  

What is the limit of $T_N$ as $N \to \infty$?

Solutions
A1. $\frac{dS_M}{dx} = \frac{-(M+1)x^M(1-x) + (1-x^{M+1})}{(1-x)^2} = \frac{-(M+1)x^M}{1-x} + \frac{S_M}{1-x}.$  

For $-1 < x < 1$, the limit does exist, and $S' = 1/(1-x)^2$.

A2. $\lim_{N \to \infty} T_N = 0.$
References


Kesten, H. 1980. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Communications in Mathematical Physics* 74: 41–59.


About the Author

Emily Puckette is an associate professor of mathematics at the University of the South in Sewanee, TN. She earned her B.A. in mathematics at Smith College and her Ph.D. at Duke University, focusing on critical probabilities and random walks. One of her favorite extracurricular activities is to go walking in the woods, pondering probabilities.