Attention: All problems are worth 10 points. Please show your work.

1. Please mark each statement as T if it is necessarily true, and F if it is not. (You do not need to show any work in this section.)

T  (F)  The parameter \( \lambda \) in a Poisson distribution is always a positive integer.

T  (F)  If \( X \) is binomially distributed, then so is \( Y = 2X \).

T  (F)  Let \( X \) denote the number of gamma particles a certain radioactive substance emits in an hour. Then \( X \) can be modeled as an exponential random variable.

T  (F)  Let \( X \) denote a random variable with mean \( \mu \), and let \( Y = g(X) \) for some monotonically increasing function \( g(x) \). Then the mean of \( Y \) is \( g(\mu) \).

T  (F)  Suppose you flip a coin 10 times and let \( X \) be the percentage of flips that landed heads. Then \( X \) is distributed binomially, with parameters \( n = 10 \) and \( p = 1/2 \).

2. Suppose you make a bet with your friend Razullo: you will roll a 6-sided die, and if the outcome is a 1 or a 6, you will pay him $4.00, and if the outcome is anything else, he will pay you $3.00. Let \( X \) denote the amount of money you win or lose. Compute the mean and variance of \( X \).

\[
\begin{align*}
X & \quad 1, 2, 3, 4, 5, 6 \\
P(X) & \quad \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \\
E(X) & \quad -4 \left( \frac{2}{6} \right) + 3 \left( \frac{4}{6} \right) \\
& \quad = \frac{2}{3} \\
\text{Var}(X) & \quad \left( -4 - \frac{2}{3} \right) \left( \frac{2}{6} \right) + \left( 3 - \frac{2}{3} \right) \left( \frac{4}{6} \right) \\
& \quad = \frac{10.9}{3}
\end{align*}
\]
3. Suppose \( X \) is a continuous random variable with density function

\[
f_X(x) = \begin{cases} 
    cx(1-x) & \text{if } 0 \leq x \leq 1 \\
    0 & \text{otherwise}
\end{cases}
\]

Find \( c \).

\[
| \int_0^1 f_X(x) \, dx | = 1.
\]

But \( \int_0^1 f_X(x) \, dx = \int_x^1 cx(1-x) \, dx \)

\[
= c \left[ \int_0^x \frac{x^2}{2} \, dx - \int_0^x \frac{x^3}{3} \, dx \right] = c \left[ \frac{x^3}{2} - \frac{x^4}{4} \right]_0^1
\]

\[
= \frac{c}{12}.
\]

Thus \( c = 12 \).

4. A point is chosen at random on a line segment of length \( L \). Find the probability that the ratio of the shorter to the longer segment is less than \( \frac{1}{4} \).

Assumption: "chosen at random" = uniformly distributed.

It is sufficient to consider points on \( [0, L] \) of line.

So let \( X \) = coordinates of chosen point. Let the ratio be \( \frac{L-x}{x} \) or \( \frac{x}{L-x} \).

\[
\text{Need } L-x < \frac{1}{4} \Rightarrow x > \frac{3}{4} L - \frac{1}{4} x \Rightarrow \frac{1}{6} x < x < \frac{L}{5}.
\]

By uniform assumption, \( P(X > \frac{L}{5}) = \frac{1}{5} \), so \( \frac{1}{6} x > \frac{1}{5} \),

\[
\frac{1}{5} x < x < \frac{L}{5}.
\]

5. Suppose \( X \) is distributed normally, with mean \( \mu = 10 \) and standard deviation \( \sigma = 2 \). Find the probability that \( X > 13 \).

\[X \sim N(10, 2)\]

\[
Z = \frac{13 - 10}{2} = 1.5
\]

By \( Z \)-table,

\[P(X < 1.5) = .93\]

\[\Rightarrow P(X > 1.5) = .07\]
6. The number of years a fruitcake lasts is exponentially distributed with parameter $\lambda = 0.1$. If Katie receives a 20 year old fruitcake this Christmas, find the conditional probability that it will still be in circulation 10 years from now. (Hint: let $X$ denote the lifespan of the fruitcake, and solve $P(X > 30 | X > 20)$.)

$$P(X > 30 | X > 20) = \frac{P(X > 30 \cap X > 20)}{P(X > 20)}$$

$$= \frac{P(X > 30)}{P(X > 20)}$$

$$= \int_{0}^{\infty} \left( \int_{30}^{\infty} e^{-\lambda x} dx \right) \frac{e^{-\lambda x}}{\int_{0}^{\infty} e^{-\lambda x} dx} dx$$

$$= \frac{e^{-3}}{e^{-2}} = e^{-1}$$

7. A pig and a goat decide to meet at a certain location. If the arrival time of the pig is uniformly distributed between noon and 1pm, while the arrival time of the goat is uniformly distributed between noon and 2pm, find the probability that the first to arrive has to wait longer than 10 minutes. (Hint: draw a picture, like we did in class.)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{600} & \text{if } 0 < x < 1 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{0}^{1} \int_{0}^{x+10} f_{X,Y}(x,y) dy dx = \int_{0}^{1} \int_{0}^{x+10} \frac{1}{600} dy dx = \int_{0}^{1} \frac{x+10}{600} dx = \frac{11}{240}$$

8. Suppose $X$ and $Y$ are independent exponential random variables, each with parameter $\lambda = 1$. Derive an expression for the density of $Z = X + Y$. (Hint: start with the cumulative distribution function.)

$$F_Z(z) = P(Z < z) = P(X + Y < z) = \int_{0}^{z} \int_{0}^{z-x} e^{-x} e^{-y} dy dx$$

$$= \int_{0}^{z} e^{-x} \left[ -e^{-y} \right]_{0}^{z-x} dx = \int_{0}^{z} e^{-x} \left[ 1 - e^{-(z-x)} \right] dx$$

$$= \int_{0}^{z} e^{-x} dx - \int_{0}^{z} e^{-2x} dx = 1 - e^{-z} - \frac{1}{2} e^{-z} = \frac{1}{2} (1 - e^{-z})$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{2} e^{-z}$$
9. The joint probability mass function of \( X \) and \( Y \) is given by

\[
p(1, 1) = 0.1 \\
p(1, 2) = 0.2 \\
p(2, 1) = 0.4 \\
p(2, 2) = 0.3.
\]

Compute the conditional mass function of \( X \) given that \( Y = i \), for \( i = 1, 2 \). Are \( X \) and \( Y \) independent?

\[
\begin{array}{c|cc}
X \times Y & 1 & 2 \\
\hline
1 & 0.1 & 0.2 \\
2 & 0.4 & 0.3 \\
\end{array}
\]

\[
\begin{array}{c|cc}
X \times Y & 1 & 2 \\
\hline
1 & 0.1 & 0.2 \\
2 & 0.4 & 0.3 \\
\end{array}
\]

10. Suppose the joint density function of \( X \) and \( Y \) is

\[
f_{X,Y}(x,y) = \begin{cases} 
  e^{-x}e^{-y} & x, y \geq 0 \\
  0 & \text{otherwise}
\end{cases}
\]

Calculate \( P(X < 2Y) \).