Markov’s inequality, Chebyshev’s inequality, and the Weak Law of Large Numbers

**Introduction:** We have spent most of the semester absorbing theory and figuring out how to apply it to concrete computational problems. It is important, however, not to conflate the practical utility of this subject with its aesthetic value as a mathematical discipline: much of what is beautiful and satisfying about probability lies in establishing approximations, bounds, inequalities, and limit theorems, often in a setting that is quite abstract and removed from any practical application.

To help you gain an appreciate for this process of “expanding the theory”, I want you to work in small groups to derive a couple of major results. We’ll spend class time this week mostly as “workshop” time. Your task is to work with your classmates to derive proofs and produce examples. You should record these proofs and examples in your notes, and be prepared to share and discuss them as a class.

**Proposition 1 (Markov’s Inequality).** If $X$ is a random variable that takes only nonnegative values, then, for any value $a > 0$,

$$P\{X \geq a\} \leq \frac{E(X)}{a}$$

**Example 1.** Produce an example (either discrete or continuous) that illustrates Markov’s Inequality.

**Proposition 2 (Chebyshev’s Inequality).** If $X$ is a random variable with finite mean $\mu$ and variance $\sigma^2$, then, for any value $k > 0$,

$$P(\{|X - \mu| \geq k\}) \leq \frac{\sigma^2}{k^2}$$

**Example 2.** Produce an example (either discrete or continuous) that illustrates Chebyshev’s Inequality.

**Proposition 3 (The Weak Law of Large Numbers).** Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables, each having finite mean $E(X_i) = \mu$ and variance $\text{Var}(X_i) = \sigma^2$. Then, for any $\epsilon > 0$,

$$P\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \epsilon\right\} \to 0 \quad \text{as} \quad n \to \infty.$$

**Example 3.** Find a way to explain what the Weak Law of Large Numbers says that will make sense to someone with little or no mathematics background.