Concept Review for Exam 3

• Double integrals
  – The double integral of \( z = f(x,y) \) over a region \( \mathcal{R} \) is the signed volume of the region between the graph of \( f \) and the region \( \mathcal{R} \).
  
  \[ \int \int_{\mathcal{R}} 1 \, dA \] can be interpreted as the area of \( \mathcal{R} \).
  
  – If \( f(x,y) \) is continuous, then the order of integration doesn’t matter (i.e. you can integrate first with respect to \( x \), then \( y \), or first with respect to \( y \), then \( x \)).
  
  – That said, if the region \( \mathcal{R} \) is \textit{vertically simple}, it is usually easier to integrate in the \( y \)-direction first, then the \( x \)-direction. If it is \textit{horizontally simple}, then it is usually easier to integrate in the \( x \)-direction first, then the \( y \)-direction.
  
  – The \textbf{average value} of \( f(x,y) \) over a region \( \mathcal{R} \) is

  \[ \overline{f} = \frac{1}{\text{Area}(\mathcal{R})} \int \int_{\mathcal{R}} f(x,y) \, dA. \]

  – If \( \mathcal{R} \) can be broken into pieces, i.e. \( \mathcal{R} = \bigcup_{i=1}^{m} \mathcal{R}_i \), then the integral of \( f(x,y) \) over \( \mathcal{R} \) can be expressed as the sum of the integrals of \( f(x,y) \) over the subregions \( \mathcal{R}_i \).

  – \textbf{Polar coordinates} are good for performing integrals over regions that exhibit circular symmetry. In polar coordinates, the \textbf{unit volume} is

  \[ dV = r \, dr \, d\theta, \]

  and the relation between \( x \) and \( y \) and \( r \) and \( \theta \) is given by

  \[
  x = r \cos \theta \\
y = r \sin \theta.
  \]

• Triple Integrals
  
  – Triples integrals are integrals over solid bodies. It is often difficult to give a simple geometric description of triple integrals.
  
  – In 3 dimensions, a \textbf{simple region} is one lying between two surfaces \( z_1(x,y) \) and \( z_2(x,y) \), all lying over some domain in the \( x - y \) plane.
  
  – As with 2-D integrals, triple integrals can, in principle, be performed iteratively, and in any order.
  
  – The \textbf{volume} of a region \( \mathcal{R} \) in 3-space is the value of the integral

  \[
  \int \int \int_{\mathcal{R}} 1 \, dV.
  \]
- The **average value** of $f(x, y, z)$ over a region $\mathcal{R}$ is

$$\overline{f} = \frac{1}{\text{Volume}(\mathcal{R})} \int \int \int_{\mathcal{R}} f(x, y, z) dV.$$ 

- **Cylindrical coordinates** are useful for integrating over regions which exhibit cylindrical symmetry.

- In cylindrical coordinates, the **unit volume** is the quantity

$$dV = r \cos \theta \, dr \, d\theta \, dz,$$

and the relation between $x$, $y$, and $z$ and $r$, $\theta$, and $\phi$ is given by

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = z.$$

- **Spherical coordinates** are good for integrating over regions which exhibit spherical symmetry.

- In spherical coordinates, the **unit volume** is the quantity

$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi,$$

and the relation between $x$, $y$, and $z$ and $r$, $\theta$, and $\phi$ is given by

$$x = \rho \cos \theta \sin \phi$$
$$y = \rho \sin \theta \sin \phi$$
$$z = \rho \cos \phi$$