Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence

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1. Introduction

Multigroup models have been introduced in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations, such as measles, mumps, gonorrhea, HIV/AIDS, West-Nile virus and vector borne diseases such as Malaria. Many factors can lead to heterogeneity in a host population. Groups can be divided geographically into communities, cities, and countries, or epidemiologically, to incorporate differential infectivity or co-infection of multiple strains of the disease agent. The seminal paper by Lajmanovich and Yorke [1] on a class of SIS multigroup models for the transmission dynamics of Gonorrhea is one of the earliest works on multigroup models. In that paper, a complete analysis of the global dynamics is established, and the proof of the global stability of the unique endemic equilibrium using a global Lyapunov function is given. Much research has been done on multigroup models in recent years as well, see, for example, [2–9] and references therein. It is well known that the global dynamics of multigroup models in high dimensions, especially the global stability of the endemic equilibrium, is a very challenging problem. The question of uniqueness and global stability of the endemic equilibrium, when the basic reproduction number $R_0$ is greater than one, has largely been open in most cases.

Recently a graph–theoretic approach to the method of global Lyapunov functions was proposed in [10–12] and it was used to establish the global stability of a unique endemic equilibrium of a multigroup SEIR model described by a system of ordinary differential equations. Their results completely solve the open problem on the uniqueness and global stability of endemic equilibrium for this class of multi-group models. By using the results or ideas of the paper [10], the uniqueness and global stability of the endemic equilibrium for several classes of multigroup epidemic models were investigated in [11–15], when the basic reproduction number $R_0$ is greater than 1, and some previously open problems were resolved.

In general, a multigroup model is formulated by dividing the population of size $N(t)$ into $n$ distinct groups. For $1 \leq k \leq n$, the $k$-th group is further partitioned into three compartments: the susceptible, infectious, and recovered, whose numbers of individuals at time $t$ are denoted by $S_k(t), I_k(t)$ and $R_k(t)$, respectively. The nonlinear term $\beta_{kj}f_{kj}(S_k, I_j)$ represents the cross infection from group $j$ to group $k$. The influx of individuals into the $k$-th group is given by a constant $\Lambda_k$, of which a fraction $p_k$ is assumed to be immune, and the remaining fraction $1 - p_k$ is susceptible. A simple immunization policy is considered...
where a fraction $\theta_k$ of the compartment $S_k$ is vaccinated. The matrix $B = (\beta_{ij})_{n \times n}$ is an irreducible contact matrix, where $\beta_{ij} \geq 0$. Within the $k$-th group, it is assumed that natural death occurs in $S_k$, $I_k$ and $R_k$ compartments with rate constants $d_k^S$, $d_k^I$ and $d_k^R$, respectively. Individuals in $I_k$ have an additional mortality due to the disease with a constant rate $\epsilon_k$. We assume that individuals in $I_k$ recover with a constant rate $\gamma_k$, and once recovered they remain permanently immune to the disease. In addition $\delta_k$ is the recovery rate of infected individuals in the $k$-th group $R_k$. Based on these assumptions, a general multigroup epidemic model with nonlinear incidence is described by the following system of differential equations:

\[
\begin{align*}
S_k' &= (1 - p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^{n} \beta_{kj} f_{ij}(S_k, I_j), \\
I_k' &= \sum_{j=1}^{n} \beta_{kj} f_{ij}(S_k, I_j) - (d_k^I + \epsilon_k + \gamma_k)I_k, \\
R_k' &= p_k\Lambda_k + \theta_k S_k + \gamma_k I_k - (d_k^R + \delta_k)R_k
\end{align*}
\]

for $1 \leq k \leq n$. The model in this form covers many previous ones in the literature, for example, \cite{10,16}.

In this paper, we consider the global dynamic behavior of the general multigroup SIR model (1). It is shown that the basic reproduction number $R_0$ (defined in Section 2) is a global threshold value in the sense that if it is less than or equal to one, the disease free equilibrium is globally asymptotically stable and the disease dies out; whereas if it is larger than one, there is a unique endemic equilibrium which is globally asymptotically stable and thus the disease persists in the population. These main results are proved in Section 2. Finally, a numerical example and simulation is also included in Section 3 to illustrate the effectiveness of the proposed result.

### 2. Main results

For each $k$, adding the three equations in (1), we obtain

\[ (S_k + I_k + R_k)' = \Lambda_k - d_k^S S_k - (d_k^I + \epsilon_k)I_k - (d_k^R + \delta_k)R_k \leq \Lambda_k - d_k^S(S_k + I_k + R_k), \]

where $d_k^S = \min(d_k^S, d_k^I + \epsilon_k, d_k^R + \delta_k)$, then

\[
\limsup_{t \to \infty} (S_k + I_k + R_k) \leq \frac{\Lambda_k}{d_k^S}.
\]

Similarly it follows from the first equation in (1) that

\[
\limsup_{t \to \infty} S_k \leq \frac{(1 - p_k)\Lambda_k}{d_k^S + \theta_k}.
\]

We assume the basic assumptions on functions $f_{ij}(S_i, I_j)$ as follows:

(H1) Define $C_{ij}(S_i) = \lim_{t \to 0^+} \frac{f_{ij}(S_i)}{\frac{1}{2}t}$. Then for all $0 < S_i \leq S_i^0$, $0 < C_{ij}(S_i) \leq \infty$, where $S_i^0 = \frac{(1 - p_i)\Lambda_i}{d_i^S + \theta_i}$;

(H2) $f_{ij}(S_i, I_j) \leq C_{ij}(S_i)$ for all $I_j > 0$ and $0 < S_i \leq S_i^0$;

(H3) $C_{ij}(S_i) < C_{ij}(S_i^0)$, for all $0 < S_i \leq S_i^0$.

Note that the class of $f_{ij}(S_i, I_j)$ satisfying (H1)–(H3) include many common incidence functionals such as $f_{ij}(S_i, I_j) = S_i I_j$ \cite{10,16}, $f_{ij}(S_i, I_j) = \mu S_i \delta_i I_j$ \cite{2}, $f_{ij}(S_i, I_j) = \frac{\mu S_i \delta_i I_j}{1 + \beta I_j}$ \cite{9}, $f_{ij}(S_i, I_j) = g(S_i) h(I_j)$ \cite{13}, $f_{ij}(S_i, I_j) = S_i \delta_i I_j$ \cite{17}, $f_{ij}(S_i, I_j) = S_i^0 I_j$ \cite{18}, $f_{ij}(S_i, I_j) = \frac{\beta S_i \delta_i I_j}{\kappa + \mu I_j}$ \cite{19}.

Since the variables $R_k$ do not appear in the first two equations of (1), we can work on the reduced system as follows:

\[
\begin{align*}
S_k' &= (1 - p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^{n} \beta_{kj} f_{ij}(S_k, I_j), \\
I_k' &= \sum_{j=1}^{n} \beta_{kj} f_{ij}(S_k, I_j) - (d_k^I + \epsilon_k + \gamma_k)I_k
\end{align*}
\]

where $k = 1, 2, \ldots, n$, in the feasible region

\[
\Gamma = \{ (S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \mathbb{R}^{2n}_+ : S_k \leq \frac{(1 - p_k)\Lambda_k}{d_k^S + \theta_k}, S_k + I_k \leq \frac{\Lambda_k}{d_k^S}, k = 1, 2, \ldots, n \}.
\]

It can be verified that $\Gamma$ in (3) is positively invariant with respect to (2). In addition, the behavior of $R_k$ can then be determined from the last equation in (1). Also let $\Gamma^o$ denote the interior of $\Gamma$.

It is clear that $P_0 = (S_0^0, 0, S_0^0, 0, \ldots, S_0^0, 0)$ is a disease-free equilibrium of the system (2), where $S_0^0 = \frac{(1 - p_i)\Lambda_i}{d_i^S + \theta_i}$. An equilibrium $P^\ast = (S_1^\ast, I_1^\ast, S_2^\ast, I_2^\ast, \ldots, S_n^\ast, I_n^\ast)$ in the interior $\Gamma^o$ of $\Gamma$ is called an endemic equilibrium, where $S_k^\ast, I_k^\ast$ satisfy the following equilibrium equations:
\[ (1 - p_k) A_k = (d_k^i + \theta_k) S_k^i + \sum_{j=1}^{n} \beta_{kj} f_{ij}(S_j^i, I_j^*) \],
\[ (d_k^i + \epsilon_k + \gamma_k) I_k^* = \sum_{j=1}^{n} \beta_{kj} f_{ij}(S_j^i, I_j^*) \].

Set \( R_0 = \rho(M_0) \) to be the spectral radius of the following matrix
\[ M_0 = M(S_0^1, S_0^2, \ldots, S_0^n) = \left( \frac{\beta_{ij} C_{ij}(S_0^i)}{d_k^i + \epsilon_i + \gamma_i} \right)_{n \times n} \).

In case that \( C_{ij}(S_0^i) = \infty \) for some \( i, j \), we set \( R_0 = \infty \). In the epidemic literature \( R_0 \) is referred to as the basic reproduction number, and our definition here is consistent with the standard ones in [20,21].

We have the following result regarding the global stability of the disease-free equilibrium:

**Theorem 2.1.** Assume that \( B = (\beta_{ij}) \) is irreducible and (H1)–(H3) hold.

1. If \( R_0 \leq 1 \), then \( P_0 \) is the unique equilibrium of the system (2) and it is globally stable in \( \Gamma \).
2. If \( R_0 > 1 \), then \( P_0 \) is unstable and the system (2) is uniformly persistent in \( \Gamma^{*} \).

**Proof.** Similar to the proof of Proposition 3.1 of [10]. Since \( B \) is irreducible, then \( M_0 \) is also irreducible. From the well-known Perron–Frobenius Theorem, \( M_0 \) has a positive principal eigenvector \( w = (w_1, w_2, \ldots, w_n) \) such that \( w_k > 0 \), \( k = 1, 2, \ldots, n \), and \( w \cdot \rho(M_0) = w \cdot M_0 \).

We define a Lyapunov function \( V = \sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} I_k^* \). Then we have
\[
V' = \sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} I_k^* = \sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} \left[ \sum_{i,j=1}^{n} \beta_{ij} C_{ij}(S_j^i) I_j - (d_k^i + \epsilon_i + \gamma_i) I_k^* \right] \\
\leq \sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} \left[ \sum_{i,j=1}^{n} \beta_{ij} C_{ij}(S_j^i) I_j - (d_k^i + \epsilon_i + \gamma_i) I_k^* \right] \\
= w \cdot (M_0 I - I) = [\rho(M_0) - 1] w \cdot I \\
\leq 0, \quad \text{if } \rho(M_0) \leq 1.
\]

Here \( I = \text{diag}(I_1, I_2, \ldots, I_n) \). If \( \rho(M_0) < 1 \), then \( V' = 0 \) if and only if \( I = 0 \). If \( \rho(M_0) = 1 \), then \( V' = 0 \) implies
\[
\sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} \left[ \sum_{i,j=1}^{n} \beta_{ij} C_{ij}(S_j^i) I_j \right] = \sum_{k=1}^{n} w_k I_k.
\]

If at least for one \( k = 1, 2, \ldots, n \), \( S_k \neq S_k^0 \), then
\[
\sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} \left[ \sum_{i,j=1}^{n} \beta_{ij} C_{ij}(S_j^i) I_j \right] < \sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} \left[ \sum_{j=1}^{n} \beta_{kj} C_{kj}(S_j^0) I_j \right] \\
= w \cdot M_0 I = w \cdot \rho(M_0) I = w \cdot I,
\]
which implies that (6) has only the trivial solution \( I = 0 \). Therefore, \( V' = 0 \) if and only if \( I = 0 \) or \( S_k = S_k^0 \) for all \( 1 \leq k \leq n \) provided that \( \rho(M_0) \leq 1 \). It can be verified that the only compact invariant subset of the set where \( V' = 0 \) is the singleton \( \{P_0\} \). Hence by LaSalle’s Invariance Principle [22], \( P_0 \) is globally asymptotically stable in \( \Gamma^{*} \) if \( \rho(M_0) \leq 1 \).

If \( R_0 = \rho(M_0) > 1 \) and \( I \neq 0 \), then
\[
w \cdot M_0 I - w = [\rho(M_0) - 1] \cdot w > 0,
\]
and thus by continuity,
\[
\sum_{k=1}^{n} \frac{w_k}{d_k^i + \epsilon_i + \gamma_i} \left[ \sum_{i,j=1}^{n} \beta_{ij} f_{ij}(S_j^i, I_j^*) - (d_k^i + \epsilon_i + \gamma_i) I_k^* \right] > 0.
\]
in a neighborhood of \( P_0 \) in \( I^\circ \). This implies that \( P_0 \) is unstable. With a uniform persistence result from [23] and a similar argument as in the proof of Proposition 3.3 of [5], the instability of \( P_0 \) implies the uniform persistence of the system (2) when \( R_0 > 1 \). This completes the proof of Theorem 2.1. \( \square \)

Now we show that the endemic equilibrium \( P_0 \) of the system (2) is unique and globally asymptotically stable when \( R_0 > 1 \). Note that the system (2) is uniformly persistent if \( R_0 > 1 \) from Theorem 2.1, together with the uniform boundedness of the solution of (2) in \( I^\circ \), then the system (2) admits at least one endemic equilibrium

\[
P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*), \quad S_i^* > 0, \quad I_i^* > 0, \quad \text{for} \ 1 \leq i \leq n.
\]

The global stability result about \( P^* \) as follow:

**Theorem 2.2.** Assume that \( B = (\beta_{ij}) \) is irreducible and \((H1)-(H3)\) hold. If \( R_0 > 1 \), \( P^* \) is an arbitrary endemic equilibrium, and \( f_k(S_k, I_j) \) satisfies the following conditions: for all \( S_k \neq S_k^*, 1 \leq k \leq n, \)

\[
(S_k - S_k^*)[f_k(S_k, I_k^*) - f_k(S_k^*, I_k^*)] > 0,
\]

and for all \( S_k, I_j > 0, 1 \leq k, j \leq n, \)

\[
(f_k(S_k^*, I_k^*)f_j(S_k, I_j) - f_j(S_k^*, I_j^*)f_k(S_k, I_k^*)) \cdot \left( \frac{f_k(S_k^*, I_k^*)f_j(S_k, I_j)}{I_j} - \frac{f_j(S_k^*, I_j^*)f_k(S_k, I_k^*)}{I_k^*} \right) \leq 0,
\]

then there exists a unique endemic equilibrium \( P^* \) for the system (2), and \( P^* \) is globally asymptotically stable in \( I^\circ \).

**Proof.** We prove that \( P^* \) is globally asymptotically stable in \( I^\circ \), which implies that the endemic equilibrium is unique. Let

\[
V_k = \int_{S_k^*}^{S_k} \frac{f_k(\xi, I_k^*) - f_k(S_k^*, I_k^*)}{f_k(S_k^*, I_k^*)} d\xi + (I_k - I_k^* \ln I_k).
\]

Then by using the equilibrium equations (4) and (5), one obtains

\[
V_k' = \left( 1 - \frac{f_k(S_k^*, I_k^*)}{f_k(S_k, I_k^*)} \right) \cdot \left[ (1 - p_k) A_k - (d_k^e + \theta_k)S_k - \sum_{j=1}^{n} \beta_{kj}f_j(S_k, I_j) \right]
+ \left( 1 - \frac{I_k}{I_k^*} \right) \cdot \left[ \sum_{j=1}^{n} \beta_{kj}f_j(S_k, I_j) - (d_k^e + \epsilon_k + \gamma_k)S_k - \sum_{j=1}^{n} \beta_{kj}f_j(S_k, I_j) \right]
= \left( 1 - \frac{f_k(S_k^*, I_k^*)}{f_k(S_k, I_k^*)} \right) \cdot \left[ (d_k^e + \theta_k)S_k + \sum_{j=1}^{n} \beta_{kj}f_j(S_k^*, I_j^*) - (d_k^e + \theta_k)S_k - \sum_{j=1}^{n} \beta_{kj}f_j(S_k, I_j) \right]
+ \left( 1 - \frac{I_k}{I_k^*} \right) \cdot \sum_{j=1}^{n} \left[ \beta_{kj}f_j(S_k, I_j) - \beta_{kj}f_j(S_k^*, I_j^*) \frac{I_k}{I_k^*} \right]
= \frac{d_k^e + \theta_k}{f_k(S_k, I_k^*)} \cdot (S_k - S_k^*)[f_k(S_k, I_k^*) - f_k(S_k^*, I_k^*)]
+ \sum_{j=1}^{n} \beta_{kj}f_j(S_k^*, I_j^*) \left[ 2 \cdot \frac{f_k(S_k^*, I_k^*)}{f_k(S_k, I_k^*)} - \frac{I_k}{I_k^*} \cdot \frac{f_k(S_k^*, I_k^*)}{f_k(S_k, I_k^*)} - \frac{I_k}{I_k^*} \cdot \frac{f_j(S_k^*, I_j^*)f_k(S_k, I_k^*)}{f_j(S_k^*, I_j^*)f_k(S_k, I_k^*)} \right].
\]

Let \( a_{ij} = \beta_{ij}(S_i^*, I_j^*) \), and

\[
F_{ij}(S_k, I_k, I_j) = 2 \cdot \frac{f_k(S_k^*, I_k^*)}{f_k(S_k, I_k^*)} - \frac{I_k}{I_k^*} \cdot \frac{f_k(S_k^*, I_k^*)}{f_k(S_k, I_k^*)} - \frac{I_k}{I_k^*} \cdot \frac{f_j(S_k^*, I_j^*)f_k(S_k, I_k^*)}{f_j(S_k^*, I_j^*)f_k(S_k, I_k^*)}.
\]

Then by condition (7),

\[
V_k' \leq \sum_{j=1}^{n} a_{ij} F_{ij}(S_k, I_k, I_j).
\]

Let \( \Phi(a) = 1 - a + \ln a \), then it is easy to verify that \( \Phi(a) \leq 0 \) for any \( a > 0 \) and the equality holds only when \( a = 1 \). Furthermore, under condition (8),
Taking $G_k(I_k) = -\frac{l_k}{I_k} + \ln \frac{l_k}{I_k}$, then we can show that $V_k, F_{kj}, G_k, a_{kj}$ satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in [12]. Therefore, the function $V = \sum_{k=1}^{n} c_k V_k$ as defined in the Theorem 3.1 of [12] is a Lyapunov function for the system (2), namely, $V' \leq 0$ for all $(S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \Gamma$. One can only show that the largest invariant subset where $V' = 0$ is the singleton $P^*$ using the same argument as in [11, 12]. By LaSalle’s Invariance Principle, $P^*$ is globally asymptotically stable in $\Gamma^\circ$. This completes the proof of Theorem 2.2. \[\square\]

We remark that Lyapunov functions for similar models have been used in [11, 13] and others. Here we take advantage of the new general result proved in [12] to prove the global stability of endemic equilibrium for a more general class of multigroup SIR models.

3. A numerical example

Consider the system (2) when $k = 2$, one has the two-group model as follows:

\[
\begin{align*}
S_1' &= (1 - p_1)A_1 - (d_1' + \theta_1)S_1 - \left[ \beta_{11} \frac{S_1 I_1}{1 + I_1} + \beta_{12} \frac{S_1 I_2}{1 + I_2} \right], \\
I_1' &= \left[ \beta_{11} \frac{S_1 I_1}{1 + I_1} + \beta_{12} \frac{S_1 I_2}{1 + I_2} \right] - (d_1' + \epsilon_1 + \gamma_1)I_1, \\
S_2' &= (1 - p_2)A_2 - (d_2' + \theta_2)S_2 - \left[ \beta_{21} \frac{S_2 I_1}{1 + I_1} + \beta_{22} \frac{S_2 I_2}{1 + I_2} \right], \\
I_2' &= \left[ \beta_{21} \frac{S_2 I_1}{1 + I_1} + \beta_{22} \frac{S_2 I_2}{1 + I_2} \right] - (d_2' + \epsilon_2 + \gamma_2)I_2.
\end{align*}
\]

(9)

Here $f_{kj}(S_k, I_j) = \frac{S_k I_j}{1 + I_j}$, for $k, j = 1, 2$. We use the parameter values as in Table 1.

| Table 1 |
| Sample values of parameters. |
| Parameter | $p_1$ | $p_2$ | $A_1$ | $A_2$ | $d_1'$ | $d_1''$ | $d_2'$ | $d_2''$ | $\theta_1$ | $\epsilon_1$ | $\gamma_1$ | $\theta_2$ | $\epsilon_2$ | $\gamma_2$ |
| Value | 1/2 | 1/3 | 2 | 3/2 | 1/4 | 1/4 | 1/8 | 1/8 | 1/4 | 3/4 | 1 | 3/4 | 3 | 7/8 |

If $B = \left( \begin{array}{cc} \frac{\beta_{11}}{\beta_{21}} & \frac{\beta_{12}}{\beta_{22}} \end{array} \right) = \left( \begin{array}{cc} 5/12 & 1/12 \\ 1/12 & 5/12 \end{array} \right)$, we have $M_0 = \left( \begin{array}{cc} 5/12 & 1/12 \\ 1/12 & 5/12 \end{array} \right)$, $R_0 = 0.5 < 1$. Hence the disease-free equilibrium $P_0 = (2, 0, 4, 0)$ is the unique equilibrium of the system (9) and it is globally stable in $\Gamma$ from Theorem 2.1.

If $B = \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right)$, we have $M_0 = \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right)$, $R_0 = 3 > 1$. Then $P^* = (0.901, 0.275, 1.065, 0.183)$ is a unique endemic equilibrium for the system (9) and it is globally asymptotically stable in $\Gamma^\circ$ from Theorem 2.2.

The numerical simulations for these two examples are shown in Fig. 1.
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