Review of Topics for Exam 3

Main topics: partial fractions; trigonometric substitution; definition of convergence for sequences; determining whether or not any given sequence converges; definition of convergence for series; determining whether or not any given series converges (integral test, comparison tests, ratio test, root test); determining the sum of geometric or telescoping series.

Partial Fractions:
Partial fraction expansions apply to rational functions of the form
\[ \frac{P(x)}{Q(x)}, \]
where \( P(x) \) and \( Q(x) \) are polynomials and the degree of \( P \) is less than the degree of \( Q \). If \( Q(x) \) factors as
\[ Q(x) = \Pi_{i=1}^{m} (x - a_i)^{\alpha_i} \cdot \Pi_{j=1}^{n} (x^2 + p_j x + q_j)^{\beta_j}, \]
then the general form for partial fraction expansion is
\[ \sum_{i=1}^{m} \frac{A_{ik}}{(x - a_i)^k} + \sum_{j=1}^{n} \frac{B_{jk} x + C_{jk}}{(x^2 + p_j x + q_j)^k}. \]
Note that the form of the expansion depends only on \( Q \), not on \( P \). (The actual coefficients will depend on \( P \), however.)
Partial fraction expansion is useful for taking integrals of rational functions, since the expansion (together with polynomial long division) reduces any rational function to functions whose form admits easy integration.

Trigonometric substitution:
Trigonometric substitution is good for solving integrals that involve \( \sqrt{x^2 - a^2} \), \( \sqrt{a^2 - x^2} \), or \( \sqrt{x^2 + a^2} \). The basic substitutions in these three cases are:
\[ x = a \sec \theta \]
\[ x = a \sin \theta \]
\[ x = a \tan \theta, \]
respectively.

Sequences:
A sequence is an (ordered) list of numbers. A sequence \( a_n \) is said to converge to a number \( L \) if and only if for every \( \epsilon > 0 \), there exists an \( N \) such that whenever \( n > N \), then
\[ |a_n - L| < \epsilon. \]
A useful tool for checking the converge of a sequence is the sandwich theorem, which states that if \( a_n \leq b_n \leq c_n \) and both \( a_n \) and \( c_n \) converge to \( L \), then so does \( b_n \). Another useful tool is L’Hospital’s Rule.

Series:
A series is the sum of a sequence. The \( n \)’th partial sum \( s_n \) of a series is defined as the sum of the first \( n \) terms of the sequence. The series converges if the partial sums converge.
Various tools allow us to assess the convergence of a series. A brief list:
1. Geometric series: (i.e. of the form \( a + ar + ar^2 + \cdots \)) These series converge if and only if \( |r| < 1 \).
   In this case, the sum is
   \[ \sum_{0}^{\infty} ar^n = \frac{a}{1 - r}. \]
2. Integral test: If \( f(x) \) is a function with \( f(n) = a_n \), then the sum of the \( a_n \)’s converges if and only if the integral of \( f(x) \) from \( c \) to \( \infty \) is finite.
3. P-series: (i.e. of the form \( \sum 1/n^p \)) These converge for \( p > 1 \), diverge for \( p \leq 1 \).
(4) **Comparison tests:** reduce convergence or divergence of $a_n$ to convergence or divergence of $b_n$, where $b_n$ is chosen to be “comparable”. Comparison tests come in several flavors. $b_n$ can be “directly comparable”, i.e. $a_n \leq b_n$ or $a_n \geq b_n$ for all $n$, or it can be “indirectly comparable”, i.e. $\lim_{n \to \infty} a_n / b_n = c$, where $0 < c < \infty$. The latter condition implies that $a_n$ and $b_n$ either both converge or both diverge. The former allows you to conclude that if the $b_n$’s diverge and $a_n \geq b_n$, then the $a_n$’s diverge, while if the $b_n$’s converge and $a_n \leq b_n$, then the $a_n$’s converge.

(5) **Ratio test:** if $\lim_{n \to \infty} a_{n+1} / a_n < 1$, then the series converges; if $\lim_{n \to \infty} a_{n+1} / a_n > 1$, then the series diverges; if $\lim_{n \to \infty} a_{n+1} / a_n = 1$, we can’t conclude anything. (Note that the ratio test does not require us to produce any $b_n$’s.)

(6) **Root test:** if $\lim_{n \to \infty} (a_n)^{1/n} < 1$, then the series converges; if $\lim_{n \to \infty} (a_n)^{1/n} > 1$, then the series diverges; if $\lim_{n \to \infty} (a_n)^{1/n} = 1$, we can’t conclude anything. (Note that the root test does not require us to produce any $b_n$’s.)