Exam 1

(1) (10 points) Let $f(x)$ be a continuous function of $x$. Explain in plain English what the following symbols mean:

(a) $\int_a^b f(x) \, dx$ It represents the area between the graph of $f(x)$ and the $x$-axis, bounded to the left and right by the lines $x=a$ and $x=b$.

(b) $\int f(x) \, dx$ It represents an antiderivative of $f(x)$.

(2) (10 points) State the two versions of the Fundamental Theorem of Calculus:

Version 1:
$$\int_a^b F(x) \, dx = F(b) - F(a),$$
where
$$\frac{dF(x)}{dx} = f(x).$$

Version 2:
$$\frac{d}{dx} \int_a^x f(u) \, du = f(x),$$
where $c$ is a constant.
(3) (10 points) Use a Riemann sum to calculate the exact value of

\[ \int_1^4 (1 - x^2) \, dx. \]

Use \( n \) rectangles.

\[ \Delta x = \frac{3}{n} \]

\( x_i = \left(1 + \frac{3i}{n}\right) \)

\( A_n = \sum_{i=0}^{n} \Delta x \cdot f(x_i) = \sum_{i=0}^{n} \frac{3}{n} \left[1 + \frac{3i}{n} + 1\right] = \sum_{i=0}^{n} \frac{3}{n} \left[1 - \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)\right] \]

\[ = -3\sum_{i=0}^{n} \frac{6i + 9i^2}{n^2} \]

\[ = \frac{18}{n^2} \sum_{i=0}^{n} i + \frac{27}{n^3} \sum_{i=0}^{n} i^2 \]

\[ = -\frac{18}{n^2} \cdot \frac{n(n+1)}{2} - \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \]

\[ = -9 + 9 = 0 \]

\[ \lim_{n \to \infty} A_n = \lim_{n \to \infty} \left[9 \left(\frac{n}{n}\right) + \frac{9}{2} \frac{n(n+1)}{n^2}\right] = -\frac{18}{2} \cdot \frac{n(n+1)}{2} - \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \]

\[ = -9 + 9 = 0 \]

(4) (10 points) Use the Fundamental Theorem of Calculus to evaluate the following:

(a) \[ \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{6}{1 + x^2} \, dx \]

\[ = 6 \arctan x \bigg|_{1/\sqrt{3}}^{\sqrt{3}} = 6 \left[\frac{\pi}{3} - \frac{\pi}{6}\right] \]

\[ = 6 \frac{\pi}{6} = \frac{\pi}{3} \]

(b) \[ \frac{d}{dx} \int_{-5}^{x} \sec^2 u \, du \]

\[ = \sec^2 (\ln x) \cdot \frac{d}{dx} \ln x = x \sec^2 (\ln x) \cdot \frac{1}{x} \]
(5) (5 points) State the Substitution Rule for Indefinite Integrals:

\[
\int f(g(x))g'(x)\,dx = \int f(u)\,du, \quad \text{where } u = g(x).
\]

(6) (15 points) Evaluate the following indefinite integrals.

(a) \( \int r^3(3-r^4)^3\,dr \)

\[
\begin{align*}
\text{Let } u &= 3-r^4 \\
\text{Then } du &= -4r^3\,dr \\
\Rightarrow \quad \frac{du}{4} &= r^3\,dr
\end{align*}
\]

\[
\Rightarrow \quad -\frac{1}{4} \int u^3\,du = -\frac{1}{4} \cdot \frac{u^6}{6} = \frac{-(3-r^4)^6}{24}
\]

(b) \( \int \cos x e^{\sin x}\,dx \)

\[
\begin{align*}
\text{Let } u &= \sin x \\
\text{Then } du &= \cos x\,dx
\end{align*}
\]

\[
\Rightarrow \quad \int e^u\,du = e^u = e^{\sin x}
\]

(c) \( \int \left(\frac{\ln x}{x}\right)^3\,dx \)

\[
\begin{align*}
\text{Let } u &= \ln x \\
\text{Then } du &= \frac{1}{x}\,dx
\end{align*}
\]

\[
\frac{1}{2} \int u^3\,du = \frac{1}{2} \cdot \frac{u^4}{4} = \frac{(\ln x)^4}{8}
\]
(7) (5 points) State the Substitution Rule for Definite Integrals:

\[ \int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du \]

(8) (15 points) Evaluate the following definite integrals.

(a) \[ \int_0^{\pi/2} t(t^2 + 1)^{1/3} \, dt \]
\[ u = t^2 + 1 = g(t) \]
\[ g(0) = 1 \]
\[ g'(t) = 2t \]
\[ du = 2t \, dt \]
\[ \Rightarrow \, \frac{du}{2} = t \, dt \]
\[ \int_0^{\pi/2} t \left( \frac{du}{2} \right) = \int_1^2 \frac{u^{1/3}}{2} \, du = \frac{3}{8} u^{4/3} \bigg|_1^2 = \frac{3}{8} \left[ 2 - 1 \right] = \frac{3}{8} \left( \frac{45}{8} \right) \]

(b) \[ \int_0^{\pi/2} \frac{\sin x}{(3 + 2 \cos x)^3} \, dx \]
\[ u = 3 + 2 \cos x = g(t) \]
\[ g(0) = 5 \]
\[ g'(x) = -2 \sin x \]
\[ du = -2 \sin x \, dx \]
\[ \Rightarrow \, \sin x \, dx = -\frac{du}{2} \]
\[ g(\frac{\pi}{2}) = 3 \]
\[ g(0) = 5 \]
\[ \int_0^{\pi/2} \frac{-\frac{1}{2} \, du}{u^3} = -\frac{1}{2} \left[ -\frac{1}{3} + \frac{1}{5} \right] = -\frac{1}{2} \left[ -\frac{8}{15} + \frac{3}{5} \right] = -\frac{1}{2} \left[ -\frac{2}{15} \right] \]

(c) \[ \int_{\pi/4}^{\pi/2} \cot x \, dx \]
\[ u = \sec t = g(t) \]
\[ g(\pi/4) = 1 \]
\[ g(\pi/2) = \sqrt{2} \]
\[ \int_{\pi/4}^{\pi/2} \frac{du}{u} = \ln u \bigg|_{1/\sqrt{2}}^{\sqrt{2}} = \ln(\sqrt{2}) - \ln(\frac{1}{\sqrt{2}}) = \frac{1}{2} \ln 2 \]
(9) **True or False:** For each of the following statements, mark T if the statement is necessarily true (i.e., always true in all circumstances), otherwise mark F. **IMPORTANT:** If you mark F, either give a counter example or explain your reasoning.

**T (F)** Suppose \( f(x) \) is a continuous function on an interval \([a, b]\) such that

\[
\int_a^b f(x) \, dx > 0.
\]

Then if \( F(x) \) is an anti-derivative of \( f(x) \), \( F(x) > 0 \) for at least some value of \( x \).

**Counter example:** \( \int_0^1 x \, dx = \frac{1}{2} \). But \( F(x) = \frac{1}{2}x - \frac{1}{2} \) is an anti-derivative of \( f(x) \), and \( F(x) < 0 \) at \( x = 1 \) on \([0, 1]\).

**T (F)** Let \( f(x) \) be continuous on the interval \([a, b]\) with \( f'(x) > 0 \) for all \( x \) and \( f(a) > 0 \). Then

\[
\int_a^b f(x) \, dx > 0.
\]

**T (F)** Suppose \( f(x) \) and \( g(x) \) are continuous functions on the interval \([a, b]\) such that

\[
\int_a^b [f(x) - g(x)] \, dx > 0.
\]

Then \( f(x) > g(x) \) for all \( x \) in \([a, b]\).

**T (F)** Suppose \( f(x) \) is a continuous function on the interval \([a, b]\) such that \( f(x) < 0 \) for all \( x \). Then

\[
\int_a^b f(x) \, dx < 0.
\]

**T (F)** Suppose \( f(x) \) is continuous on \([a, b]\) and

\[
\int_a^b f(x) \, dx < 0.
\]

Then any finite sum approximation of \( \int_a^b f(x) \, dx \) will also be negative.
(10) (5 points) Relax. You have only one question left, and it's an easy one. Draw a picture of a dancing bear to celebrate.

(11) (5 points) Fill in the following chart:

<table>
<thead>
<tr>
<th>θ</th>
<th>π/6</th>
<th>π/4</th>
<th>π/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin θ</td>
<td>1/2</td>
<td>1/√2</td>
<td>√3/2</td>
</tr>
<tr>
<td>cos θ</td>
<td>1/√3</td>
<td>1/√2</td>
<td>1/2</td>
</tr>
<tr>
<td>tan θ</td>
<td>1/√3</td>
<td>1</td>
<td>√3</td>
</tr>
</tbody>
</table>

What is arcsin(1/2)? \( π/6 \)