2 The Gauss-Newton algorithm

This section provides a concise summary of the results in [1] relevant to what will follow. It is intended as a reference, and thus omits proofs and details. The reader interested in these matters should consult the original paper.

2.1 The algorithm

The Gauss-Newton method [3] can be brought to bear on this problem as follows: define a state vector \( S \) as

\[
S = [s^*; s_1; \ldots; s_n],
\]

where

\[
s^* := [a_1, \ldots, a_{m_u}, b_1, \ldots, b_{m_v}, c_1, \ldots, c_{m_w}]^T \tag{7}
\]

and the \( s_i \) are as in (4). Define an observation vector \( O \) as

\[
O := [u_1^1, v_1^1, w_1^1, \ldots, u_1^n, v_1^n, w_1^n].
\]

If \( F : \mathbb{R}^{m_u+m_v+m_w+6n} \rightarrow \mathbb{R}^{3mn} \) is the forward map taking each \( S \) to a unique \( O \), the least squares solution \( \hat{S} \) is

\[
\hat{S} := \arg \min_S \| F(S) - O \|^2.
\]

The Gauss-Newton algorithm finds \( \hat{S} \) by solving a sequence of least squares equations of the form

\[
J(S_k)^T J(S_k) \delta S_k = -J(S_k)^T (F(S_k) - O), \tag{8}
\]

where \( S_0 \) is an initial guess, \( S_k \) is the \( k \)-th update to that guess, \( J(S_k) \) is the Jacobian of \( F \) at \( S_k \), and \( \delta S_k \) satisfies \( S_{k+1} = S_k + \delta S_k \).

2.2 The update step

Since the \( 3m \) observations taken at a given probe placement have no bearing on the position parameters for any other placement, the observation vector can be decomposed into pieces

\[
o_i := (o^1_i; \ldots; o^{m_i}_i), \quad o^j_i = (u^j_i, v^j_i, w^j_i)^T, \tag{9}
\]

whereupon the forward map can be written as stacked vectors \( f(s^*, s_i) \), where

\[
f(s^*, s_i) = o_i, \quad i = 1 \cdots n.
\]

With this notation, the least squares estimate becomes

\[
\hat{S} := \arg \min_S \sum_i \| f(s^*, s_i) - o_i \|^2.
\]

As a consequence, the Jacobian \( J \) is sparse, with a general form

\[
J = \begin{bmatrix}
U_1 & V_1 & 0 & \cdots & \cdots & 0 \\
U_2 & 0 & V_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & 0 & \cdots & \vdots \\
U_n & 0 & \cdots & \cdots & 0 & V_n
\end{bmatrix}, \tag{10}
\]