Homework 1: Difference Equations

The following sequence of (loosely) related problems should be done in groups. Each group will submit a single write-up, and every member of the group will get the same grade. It thus behooves the group members to figure out how to work collaboratively and share the work burden.

The write-up should be typed, ideally in latex. Attention should be paid to complete sentences and proper grammar. Mathematical arguments can be informal, but should be correct, and complete in the sense of capturing all key ideas. Qualitative conclusions should say something interesting not just about the mathematics but about the underlying process.

(Age-structured populations) One of the problem with the population models we studied in class is that they are too simple: they capture gross population level dynamics, but omit details about which sub-populations are reproducing and which are not. One way to include such details is to break the population into age classes, i.e. subgroups of fixed ages, each of which exhibits distinct reproductive behavior. This problem, which is based on problem 1.20 in Mathematical Models in Biology, explores a methodology for understanding age-structured linear difference models.

A. Suppose the total population in generation \( n \) has \( m \) age classes, each of size \( p_i^n \). Let \( \alpha_i, i = 1 \cdots m \) denote the number of births from individuals of a given age class, and \( \sigma_i, i = 1 \cdots m - 1 \) denote the fraction of \( i \) year olds that survive to be \( i + 1 \) years old. Show that the population can be described by the following matrix equation:
\[
P_{n+1} = AP_n,
\]
where
\[
P_n = \begin{pmatrix}
p_1^n \\
p_2^n \\
\vdots \\
p_m^n
\end{pmatrix}, \quad A = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_m \\
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
& & & \\
0 & \cdots & \sigma_{m-1} & 0
\end{pmatrix}
\]
The matrix \( A \) is called a Leslie matrix.

B. Note that this equation is an \( m \) dimensional system of first order difference equations. It is a fact that every \( m \) dimensional first order system corresponds to a unique \( m \)th order equation in one variable, and vice versa. Prove this for the case \( m = 2 \). Also, show that the characteristic polynomial of the matrix that defines the 2 dimensional system is the same as the characteristic polynomial of the corresponding second order equation in one variable. (Recall that the characteristic polynomial for a matrix is given by \( \det(\lambda I - A) \).)

C. Show that the solutions to the system in part A) are roots of the polynomial:
\[
\lambda^m - \alpha_1 \lambda^{m-1} - \alpha_2 \sigma_1 \lambda^{m-2} - \alpha_3 \sigma_1 \sigma_2 \lambda^{m-3} - \cdots - \alpha_m \sigma_1 \sigma_2 \cdots \sigma_{m-1} = 0.
\]
D. Show that the matrix $A$ in part A) has exactly one positive eigenvalue, $\lambda^*$. (Hint: show that the function

$$f(\lambda) \equiv 1 - \frac{p_m(\lambda)}{\lambda^m}$$

decreases monotonically for $\lambda \in (0, \infty)$, going to $\infty$ as $\lambda \to 0$ and going to 0 as $\lambda \to \infty$. (Here, $p_m$ refers to the characteristic polynomial.) Use these observations to prove the assertion.

E. Suppose $\vec{v}^\ast$ is the eigenvector corresponding to $\lambda^*$, and that $|\lambda^*|$ is strictly greater than $|\lambda|$ for any other eigenvalue $\lambda$. Show that in the limit as $n \to \infty$, the age distribution mirrors the proportions in $\vec{v}^\ast$.

F. Gorilla reproductive intervals closely mimic those of humans. In particular, if the population of female gorillas is divided into three age class, 0-15, 15-30, and 30-45, then a Leslie model for the population dynamics is given by

$$
\begin{pmatrix}
  x_1(n + 1) \\
  x_2(n + 1) \\
  x_3(n + 1)
\end{pmatrix}
= 
\begin{pmatrix}
  0.43 & 0.85 & 0.13 \\
  0.99 & 0 & 0 \\
  0 & 0.98 & 0
\end{pmatrix}
\begin{pmatrix}
  x_1(n) \\
  x_2(n) \\
  x_3(n)
\end{pmatrix},
$$

where $x_i(n)$ represents the size of the $i$th age class in generation $n$. Use this model to draw both qualitative and quantitative conclusions about the rate at which the gorilla populations grows or shrinks, and what the long-term distribution among age-classes will be. Call attention to any obvious limitations of the model.