Solution by Jesse Jenks
Problem 4.62

**Problem Statement:** If $f$ is one-to-one and continuous on $[a, b]$, then $f$ is strictly monotonic.

Assume for the sake of contradiction that $f$ is not monotonic. Then there are points $x, y, z$ in $[a, b]$, so that $x < y < z$ and $f(y) \geq f(x)$ and $f(z) \leq f(y)$ or $f(y) \leq f(x)$ and $f(z) \geq f(y)$. Note that $f(y)$ is not equal to $f(x)$ or $f(z)$ since $f$ is one-to-one. Now we can assume without loss of generality that $f(y) > f(x)$ and $f(z) < f(y)$. Then the intersection $[f(x), f(y)] \cap [f(z), f(y)]$ is a nonempty interval. Let $c$ be a point in this intersection distinct from $f(y)$. Since $f$ is real valued and continuous on the compact intervals $[x, y]$ and $[y, z]$, by the intermediate value theorem, there are points $c_1 \in [x, y]$ and $c_2 \in [y, z]$ such that $f(c_1) = c$ and $f(c_2) = c$, but $c_1 \neq c_2$. However, this contradicts the fact that $f$ is one-to-one, so $f$ must be monotonic. To see that $f$ is strictly monotonic, note that since $f$ is one-to-one, $f(x) \neq f(y)$ for any distinct $x, y$ in $[a, b]$. 