Problem Statement:

Find the flaw in the following “proof” that the set of all intervals of positive length is countable.

Let \( \{x_1, x_2, \ldots \} \) denote the countable set of rational numbers and let \( I \) be any interval of positive length. Then \( I \) contains infinitely many rational points \( x_n \), but among these there will be one with smallest index \( n \). Define a function \( F \) by means of the equation \( F(I) = n \), if \( x_n \) is the rational number with smallest index in the interval \( I \). This function establishes a one-to-one correspondence between the set of all intervals and a subset of the positive integers. Hence the set of all intervals is countable.

The flaw in the proof lies in the second to last sentence. Although the function \( F \) is well defined, it is not one-to-one. For example, suppose the rational number \( \frac{1}{2} \) has index 3, and all other rationals in the interval \( [0,1] \) have a higher index. In this case \( F([0,1]) = 3 \). But notice that \( \frac{1}{2} \) must also have the smallest index in \( [\frac{1}{4}, \frac{3}{4}] \), since \( [\frac{1}{4}, \frac{3}{4}] \subset [0,1] \). Then \( F([\frac{1}{4}, \frac{3}{4}]) = 3 \), so \( F \) is not one-to-one and does not establish that the set of all intervals is countable. However, it does not mean the set of all intervals is uncountable either, just that the conclusion drawn from the definition of \( F \) is incorrect.