Solution by Hayden Harper

Problem 2.2

**Problem Statement:** Let $S$ be a relation and $\mathcal{D}(S)$ be its domain. A relation which is symmetric, reflexive, and transitive is called an equivalence relation. Determine which of these properties is possessed by $S$, if $S$ is the set of all pairs of real numbers $(x, y)$ such that

\[
\begin{align*}
a) & \ x \leq y, \\
b) & \ x < y, \\
c) & \ x < |y|, \\
d) & \ x^2 + y^2 = 1, \\
e) & \ x^2 + y^2 < 0, \\
f) & \ x^2 + x = y^2 + y.
\end{align*}
\]

a) $S$ is the set of all pairs of real numbers $(x, y)$ such that $x \leq y$.

**Proof.** Here, $\mathcal{D}(S) = \mathbb{R}$. Let $x \in \mathcal{D}(S)$. Then clearly, $x \leq x$, so $(x, x) \in S$. Hence $S$ is reflexive. Now, note that $(5, 7) \in S$ since $5 \leq 7$ and $5, 7 \in \mathbb{R}$. However, $7 \not\leq 5$, so $(7, 5) \notin S$, and thus $S$ is not symmetric by counterexample. Next, let $(x, y) \in S$ and $(y, z) \in S$. Then $x \leq y$ and $y \leq z$. Hence, $x \leq y \leq z$, so $(x, z) \in S$ and $S$ is transitive. Since $S$ is only reflexive and transitive, it is not an equivalence relation.

b) $S$ is the set of all pairs of real numbers $(x, y)$ such that $x < y$.

**Proof.** Here, $\mathcal{D}(S) = \mathbb{R}$. Let $x \in \mathcal{D}(S)$. Then clearly, $x \not< x$, so $(x, x) \notin S$. Hence $S$ is not reflexive. Now, note that $(5, 7) \in S$ since $5 < 7$ and $5, 7 \in \mathbb{R}$. However, $7 \not< 5$, so $(7, 5) \notin S$, and thus $S$ is not symmetric by counterexample. Next, let $(x, y) \in S$ and $(y, z) \in S$. Then $x < y$ and $y < z$. Hence, $x < y < z$, so $(x, z) \in S$ and $S$ is transitive. Since $S$ is only transitive, it is not an equivalence relation.

c) $S$ is the set of all pairs of real numbers $(x, y)$ such that $x < |y|$.

**Proof.** Here, $\mathcal{D}(S) = \mathbb{R}$. Since $0 \in \mathbb{R}$ but $0 \not< |0|$, $S$ is not reflexive by counterexample. Now, note that $(-5, 7) \in S$ since $-5 < |7| = 7$ and $-5, 7 \in \mathbb{R}$. However, $7 \not< |-5| = 5$, so $(7, -5) \notin S$, and thus $S$ is not symmetric by counterexample. Next, note that $(0, -1) \in S$ since $0 < |-1| = 1$, and $(-1, 0) \in S$ since $-1 < |0| = 0$, but $(0, 0) \notin S$ as demonstrated above. Hence, $S$ is not transitive by counterexample. Since $S$ possesses none of these properties, $S$ is certainly not an equivalence relation.

d) $S$ is the set of all pairs of real numbers $(x, y)$ such that $x^2 + y^2 = 1$.

**Proof.** Here, $\mathcal{D}(S) = [-1, 1]$. Since $1 \in \mathcal{D}$ but $1^2 + 1^2 \neq 1$, $(1, 1) \notin S$, so $S$ is not reflexive by counterexample. Let $(x, y) \in S$. Then $x^2 + y^2 = 1$. By the commutative property of addition of real numbers, $y^2 + x^2 = 1$. Then $(y, x) \in S$ so $S$ is symmetric. Next, note that $(1, 0) \in S$ since $1^2 + 0^2 = 1$, and $(0, 1) \in S$ since $0^2 + 1^2 = 1$, but $(1, 1) \notin S$ as demonstrated above. Hence, $S$ is not transitive by counterexample. Since $S$ is only symmetric, $S$ is not an equivalence relation.

e) $S$ is the set of all pairs of real numbers $(x, y)$ such that $x^2 + y^2 < 0$.

**Proof.** Here, $\mathcal{D}(S) = \emptyset$. Thus, $S$ is transitive, symmetric, and reflexive since it satisfies those properties for all the points that are in $S$. Hence, $S$ is an equivalence relation.

f) $S$ is the set of all pairs of real numbers $(x, y)$ such that $x^2 + x = y^2 + y$.

**Proof.** Here, $\mathcal{D}(S) = \mathbb{R}$. If $x \in \mathbb{R}$, then clearly $x^2 + x = x^2 + x$ so $(x, x) \in S$ and $S$ is reflexive. Now, let $(x, y) \in S$. Then $x^2 + x = y^2 + y$. Clearly, $y^2 + y = x^2 + x$, so $(y, x) \in S$ and $S$ is symmetric. Next, let $(x, y) \in S$ and $(y, z) \in S$. Then $x^2 + x = y^2 + y$ and $y^2 + y = z^2 + z$. Then $x^2 + x = y^2 + y = z^2 + z$, so $(x, z) \in S$ and $S$ is transitive. Having satisfied these three properties, $S$ is an equivalence relation.