Problem Statement: Show that the following sets are countable: a) the set of circles in the complex plane having rational radii and centers with rational coordinates, b) any collection of disjoint intervals of positive length.

a) The set of circles in the complex plane having rational radii and centers with rational coordinates is countable.

Proof. Let $C$ be the set of circle in the complex plane having rational radii and centers with rational coordinates. Define $F : C \to \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+$ by $F(c) = (x_c, y_c, r_c)$ for $c \in C$, where $(x_c, y_c)$ is the rational center of $c$ and $r_c$ is the positive rational radius of $c$. Clearly, $F$ is injective. Hence, $C \sim \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+$ by definition. Since $\mathbb{Q}$ is countable, $\mathbb{Q}^+$ is countable by theorem 2.16. Since the countable Cartesian product of countable sets is countable, $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}^+ \sim \mathbb{Z}^+$. By the transitivity of set equinumerosity, $C \sim \mathbb{Z}^+$, so $C$ is countable.

b) Any collection of disjoint intervals of positive length is countable.

Proof. Let $S$ be a collection of disjoint intervals of positive length. Let $Q = \{x_1, x_2, \cdots \}$ denote the countable set of rational numbers. Define $F : S \to \mathbb{Z}^+$ by $F(I) = n$, where $I \in S$ and $n = \min\{j : x_j \in Q \cap I\}$. To show that $F$ is injective, suppose that $I_1 \neq I_2$, and $F(I_1) = F(I_2) = n$. Then $F(I_1) = n$ implies that $x_n \in I_1$, and $F(I_2) = n$ implies that $x_n \in I_2$. Thus, $x_n \in I_1 \cap I_2 = \emptyset$. Hence, $I_1 = I_2$, so $F$ is injective. Therefore, $S$ is similar to a subset of $\mathbb{Z}^+$, so by theorem 2.16 and the transitivity of set equinumerosity, $S$ is countable.