Problem 2.13

**Problem Statement:** Prove that every infinite set $S$ contains a proper subset similar to $S$.

*Proof.* Let $S$ be an infinite set and let $A$ be a countable subset of $S$ where $A = \{x_1, x_2, ..., x_n\}$ such that $S = \tilde{S} \cup A$ and $\tilde{S} = S - A$. Define,

\[ \phi : A \rightarrow A \]
\[ x_i \rightarrow x_{i+1}. \]

Note that $\phi(A) \subset A$, but $\phi(A) \neq A$ since $x_1 \notin \phi(A)$. Thus, $\phi(A)$ is a proper subset of $A$ and $\tilde{S} \cup \phi(A)$ is a proper subset of $S$. Finally, define the map

\[ \delta : S \rightarrow \tilde{S} \cup \phi(A) \]

as,

\[ \delta(x) = \begin{cases} 
  x & \text{if } x \in \tilde{S} \\
  \phi(A) & \text{if } x \in A.
\end{cases} \]

Note that $\delta(S) = \tilde{S} \cup \phi(A) \subset S$ and that $\delta$ is injective. Thus, delta is a bijective mapping onto its image, and this bijection establishes that $S$ and $\tilde{S} \cup \phi(A)$ are similar. This concludes the proof. \[\square\]